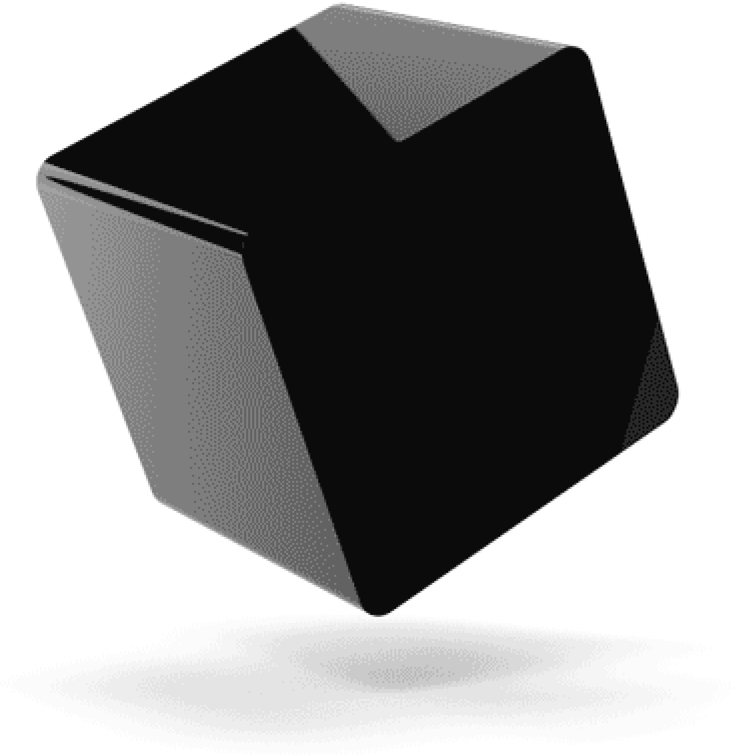
GRK 8

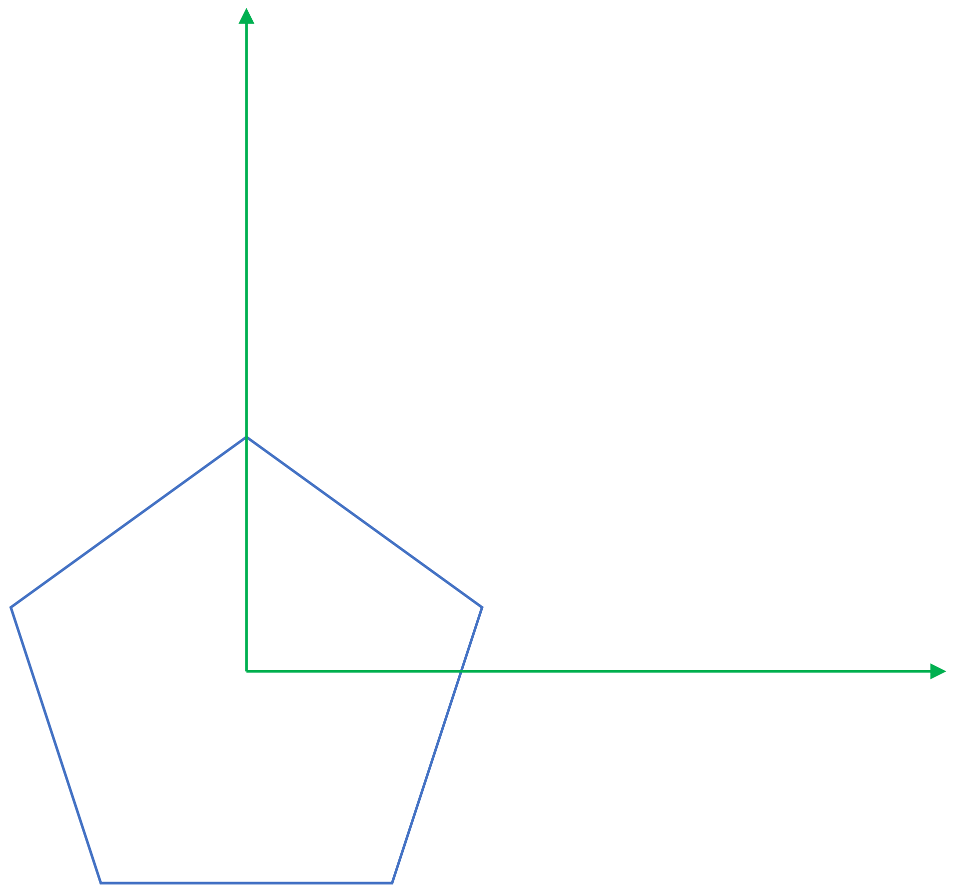
Dr Wojciech Palubicki

Rotation



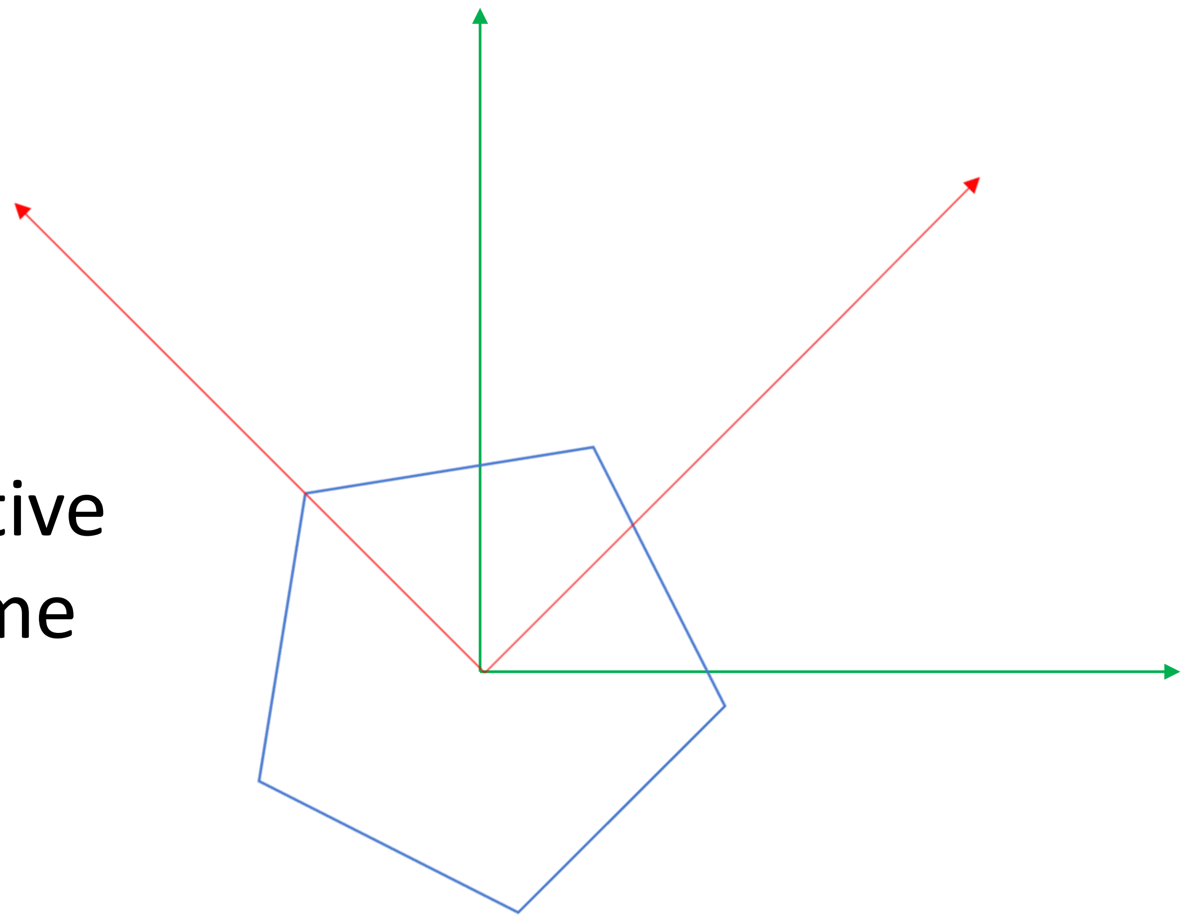
Defining Rotation

Reference frame



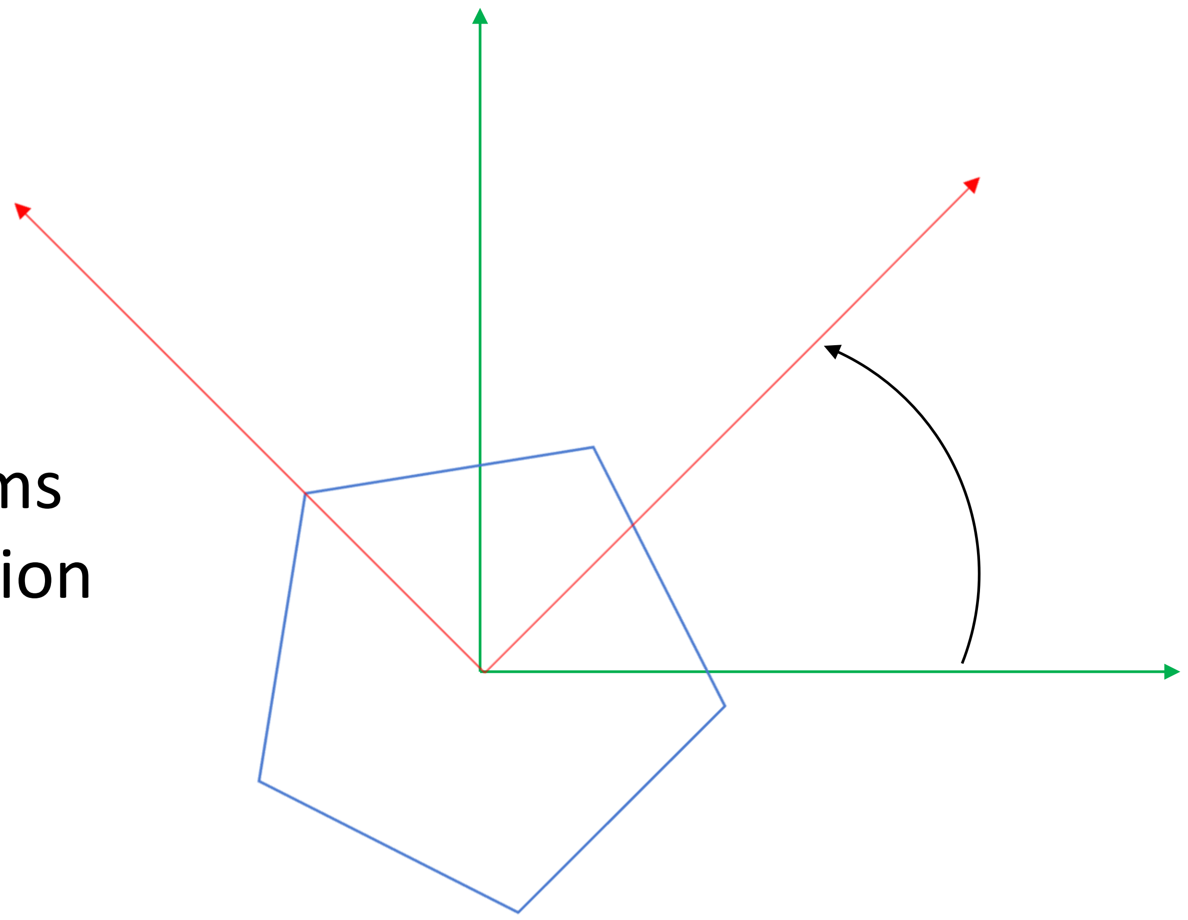
Defining Rotation

Orientation relative to reference frame

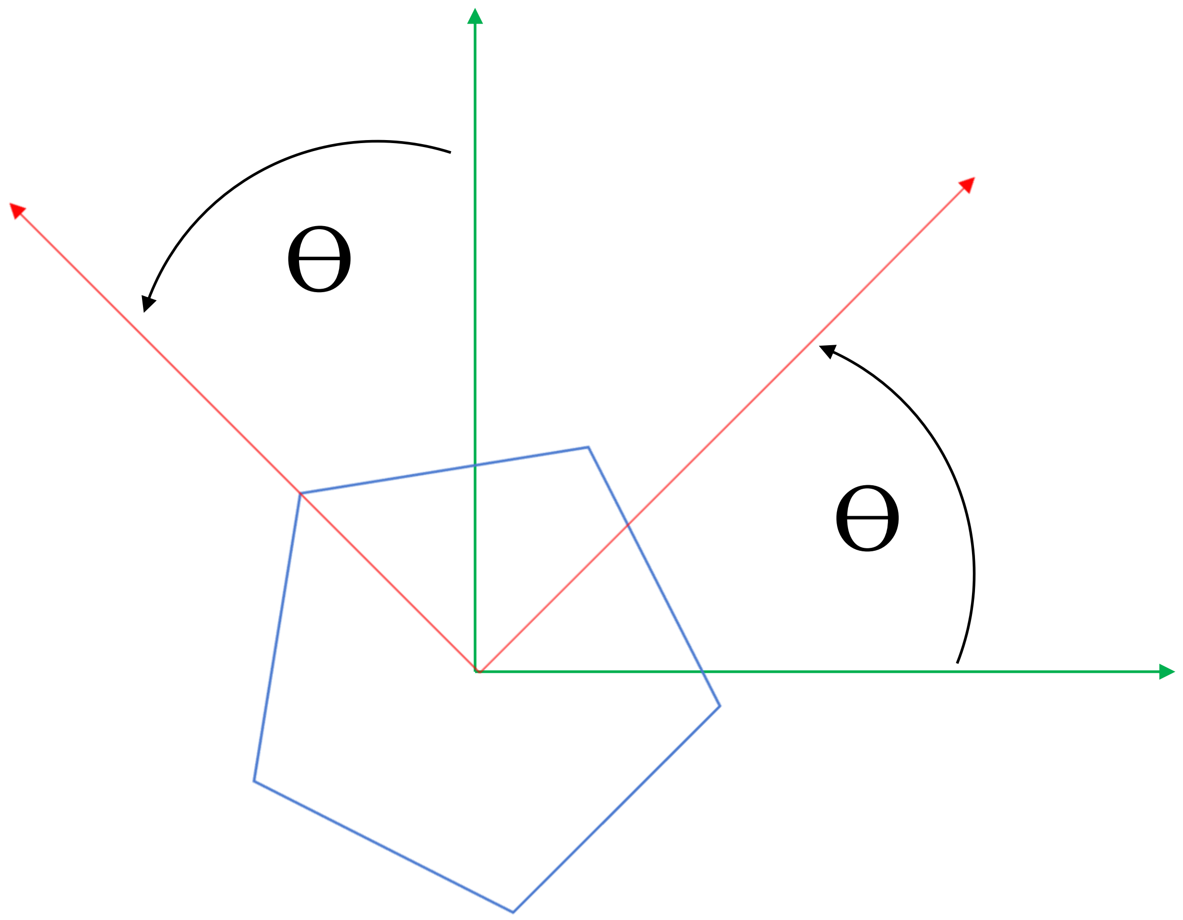


Defining Rotation

Rotation transforms from one orientation to another



Rotation Angle (2D)



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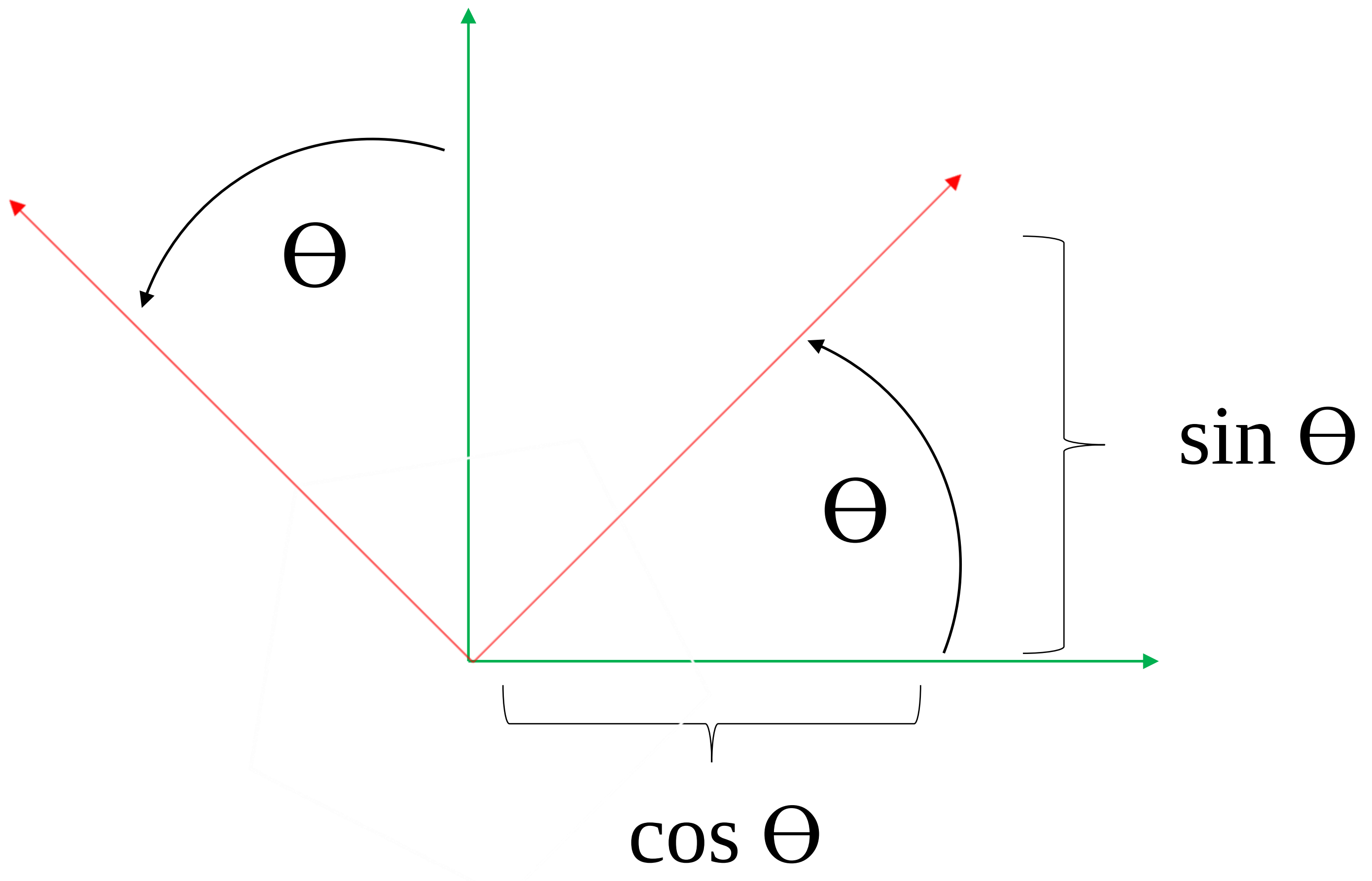

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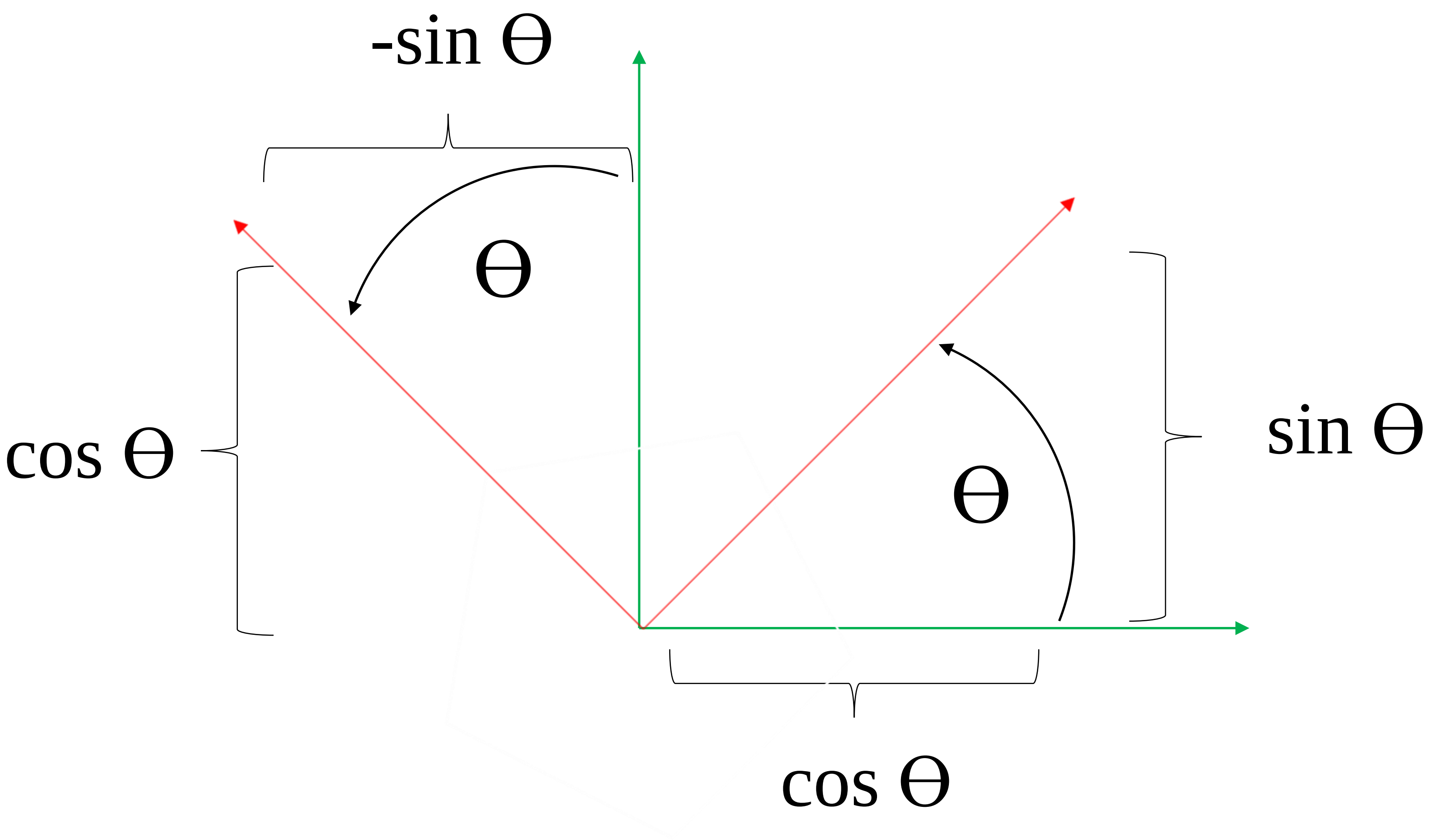

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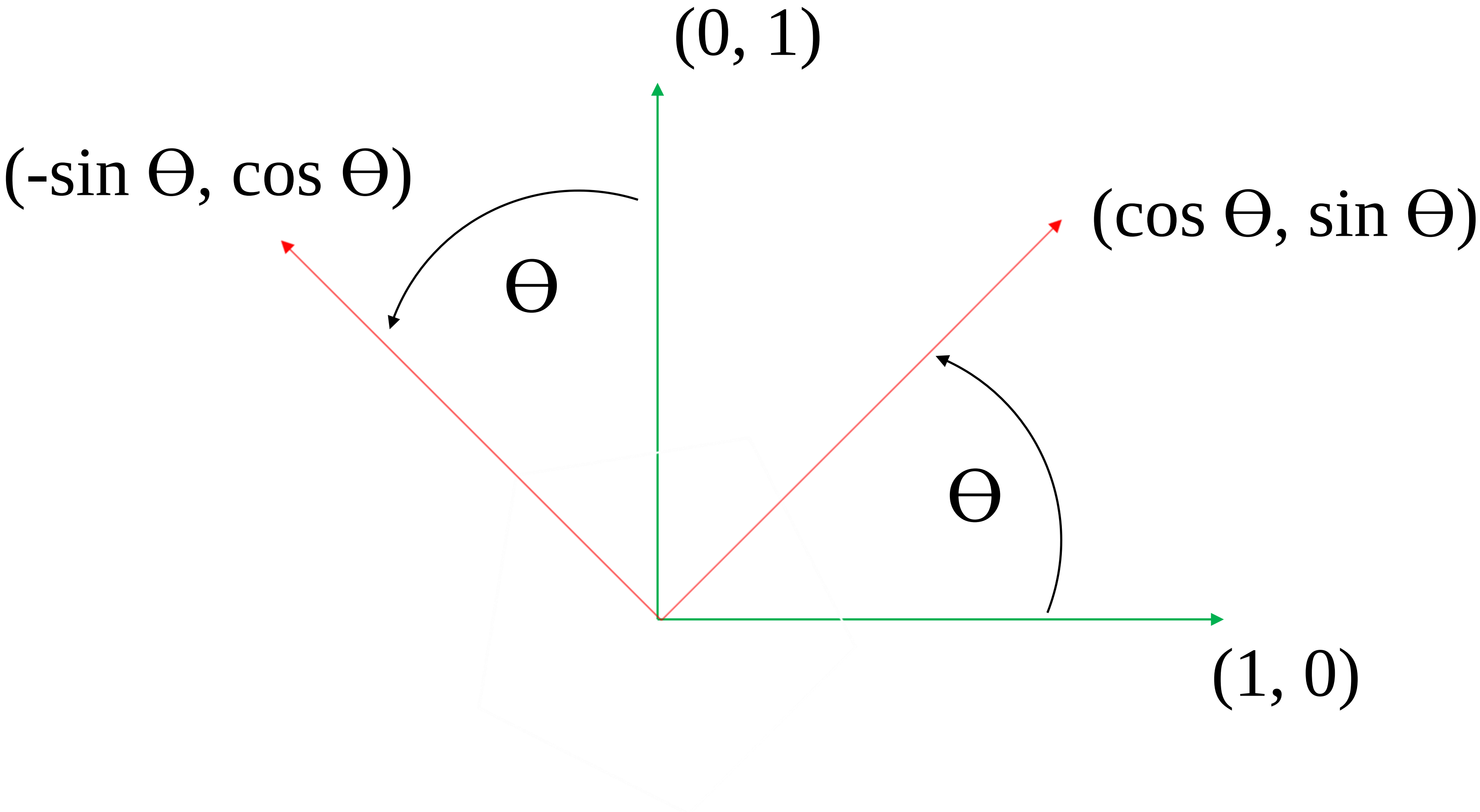
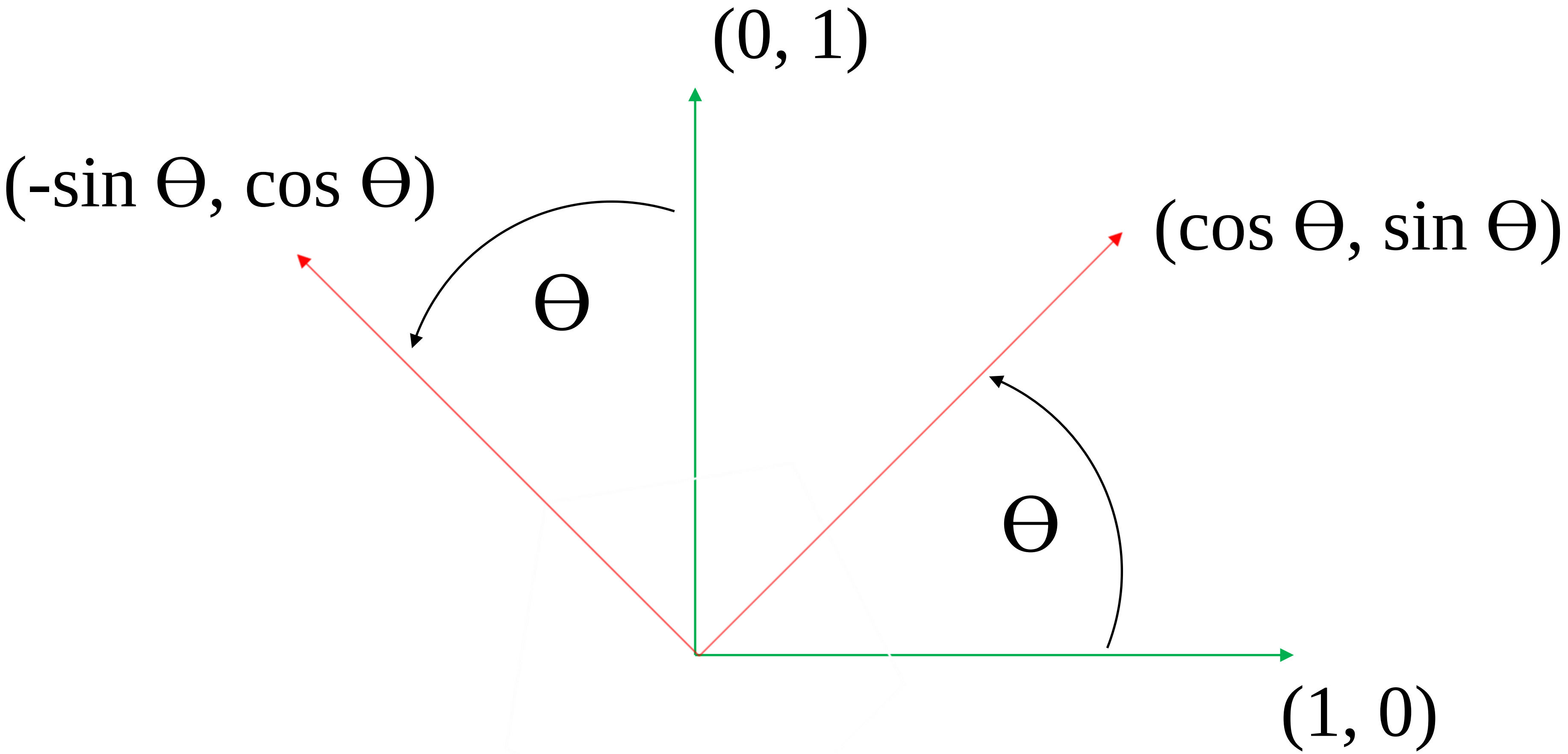


Transformation



$(x, y) \rightarrow (x \cos \Theta - y \sin \Theta, x \sin \Theta + y \cos \Theta)$

Orientation vs Rotation

- **Orientation** of an object is described relative to some reference frame
- A **rotation** changes an object from one **orientation** to another

2D Rotation Matrix

- Transformation
  - (1, 0) → (cos Θ, sin Θ)
  - (0, 1) → (-sin Θ, cos Θ)

- Rotation matrix
  - $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

3D Rotation Matrices

- Rotation Matrix attributes:
  - Orthogonal
    - Basis vectors are unit length
    - Dot products of basis vectors are zero
    - $M^{-1} = M^T$
    - $M^T M = \mathbf{I}$
  - Determinant is 1

3D Matrices $R_x R_y R_z$

$$R_X R_Y R_Z$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos A & -\sin A \\ 0 & \sin A & \cos A \end{pmatrix} \begin{pmatrix} \cos B & 0 & \sin B \\ 0 & 1 & 0 \\ -\sin B & 0 & \cos B \end{pmatrix} \begin{pmatrix} \cos C & -\sin C & 0 \\ \sin C & \cos C & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3D Matrices $R_x R_y R_z$

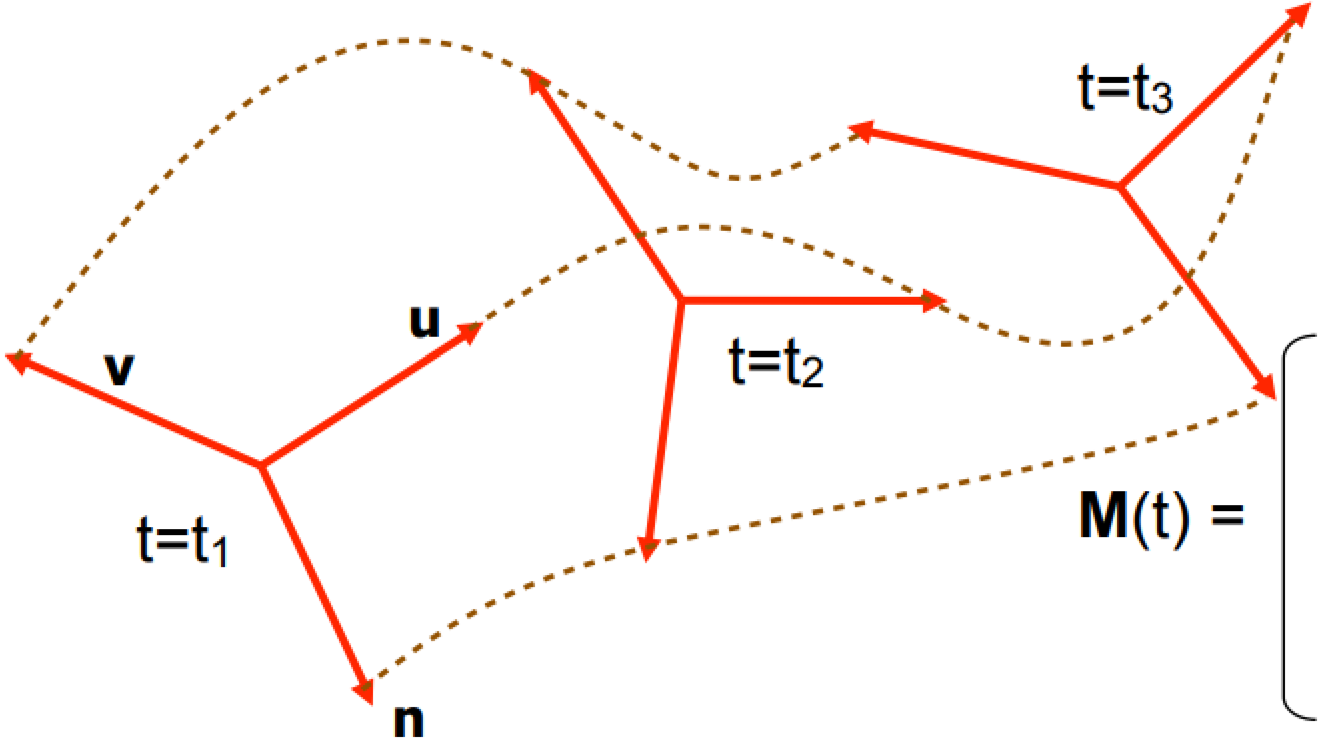
$$
\begin{aligned}
&R_X R_Y R_Z \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos A & -\sin A \\ 0 & \sin A & \cos A \end{pmatrix} \begin{pmatrix} \cos B & 0 & \sin B \\ 0 & 1 & 0 \\ -\sin B & 0 & \cos B \end{pmatrix} \begin{pmatrix} \cos C & -\sin C & 0 \\ \sin C & \cos C & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos B & 0 & \sin B \\ \sin A \sin B & \cos A & -\sin A \cos B \\ -\cos A \sin B & \sin A & \cos A \cos B \end{pmatrix} \begin{pmatrix} \cos C & -\sin C & 0 \\ \sin C & \cos C & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos B \cos C & -\cos B \sin C & \sin B \\ \sin A \sin B \cos C + \cos A \sin C & -\sin A \sin B \sin C + \cos A \cos C & -\sin A \cos B \\ -\cos A \sin B \cos C + \sin A \sin C & \cos A \sin B \sin C + \sin A \cos C & \cos A \cos B \end{pmatrix}
\end{aligned}
$$

Pitch, Yaw, Roll

Ideal Orientation Format

- Represent 3 degrees of freedom with minimum number of values
- Allow concatenations of rotations
- Math should be simple and efficient
  - concatenation
  - rotation
  - interpolation

Interpolation?



$$\mathbf{M}(t) = \begin{bmatrix} u_x(t) & u_y(t) & u_z(t) \\ v_x(t) & v_y(t) & v_z(t) \\ n_x(t) & n_y(t) & n_z(t) \end{bmatrix}$$

Interpolating Matrices

- Say we interpolate halfway between each element

$$0.5\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}+0.5\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}0.5 & -0.5 & 0\\ 0.5 & 0.5 & 0\\ 0 & 0 & 1\end{pmatrix}$$

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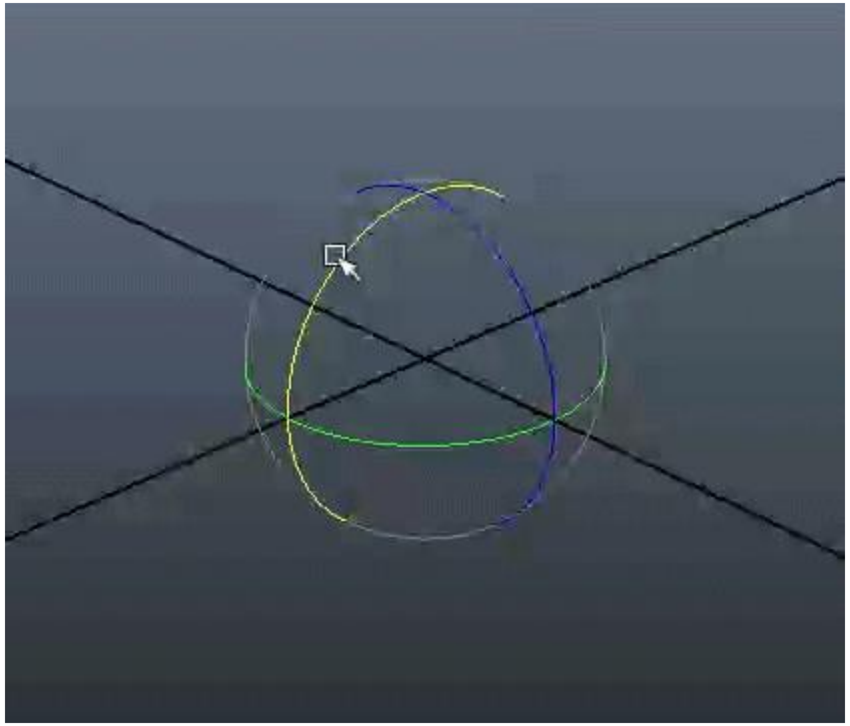
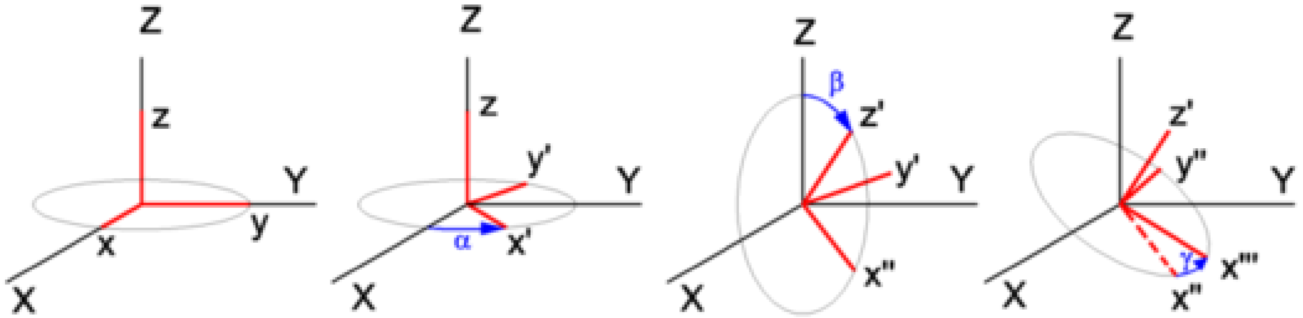
- Result isn't a rotation matrix!
- Need Gram-Schmidt orthonormalization
- → Interpolation computationally costly with rotation matrices

Gimbal Lock



Euler Angles



Euler Angles

- Concatenation (yes, with matrices – somewhat difficult)
- Rotation (Gimbal lock problem)
- Interpolation (no, with matrices)

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ψ

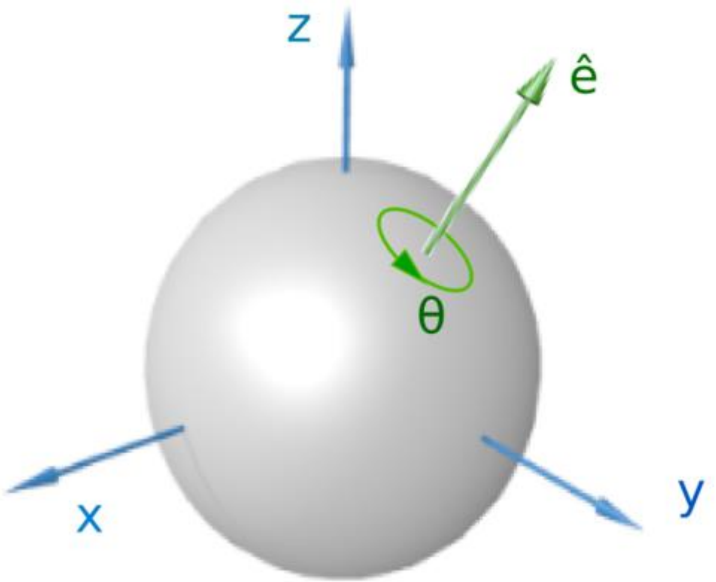
θ

$$\mathbf{R} = \mathbf{R}_Z(\psi)\mathbf{R}_Y(\theta)\mathbf{R}_X(\phi)$$
$$= \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$
$$= \begin{pmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

$$\phi = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$
$$\psi = -\mathrm{Atan2}\left(\frac{r_{21}}{\cos(\phi)}, \frac{r_{11}}{\cos(\phi)}\right)$$
$$\theta = \mathrm{Atan2}\left(\frac{r_{32}}{\cos(\phi)}, \frac{r_{33}}{\cos(\phi)}\right)$$

Euler’s Rotation Theorem

- Every 3D rotation can be represented by an axis and an angle.

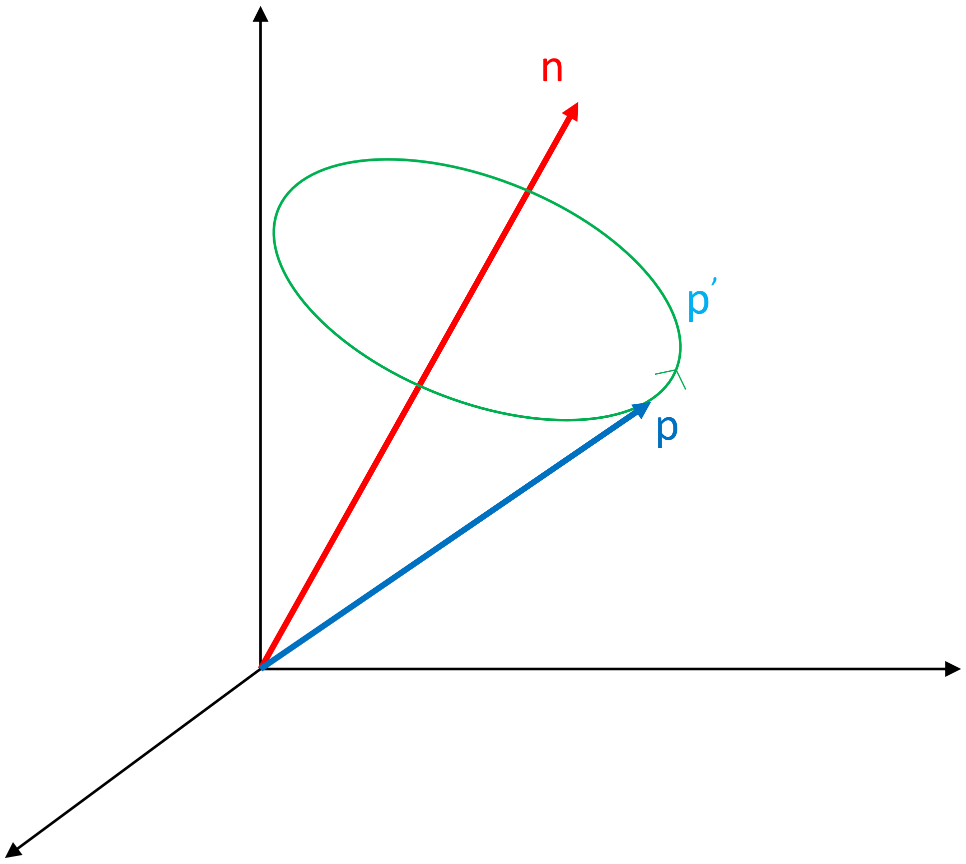


Rodrigues Rotation Theorem

- Given axis $\hat{r}$, angle θ and point p rotation is

$$\mathrm{R}(\hat{\mathbf{r}}, \theta, \mathbf{p}) = \mathbf{p}\cos\theta + (\hat{\mathbf{r}} \times \mathbf{p})\sin\theta + \hat{\mathbf{r}}(\hat{\mathbf{r}} \bullet \mathbf{p})(1 - \cos\theta)$$

Rodrigues Formula



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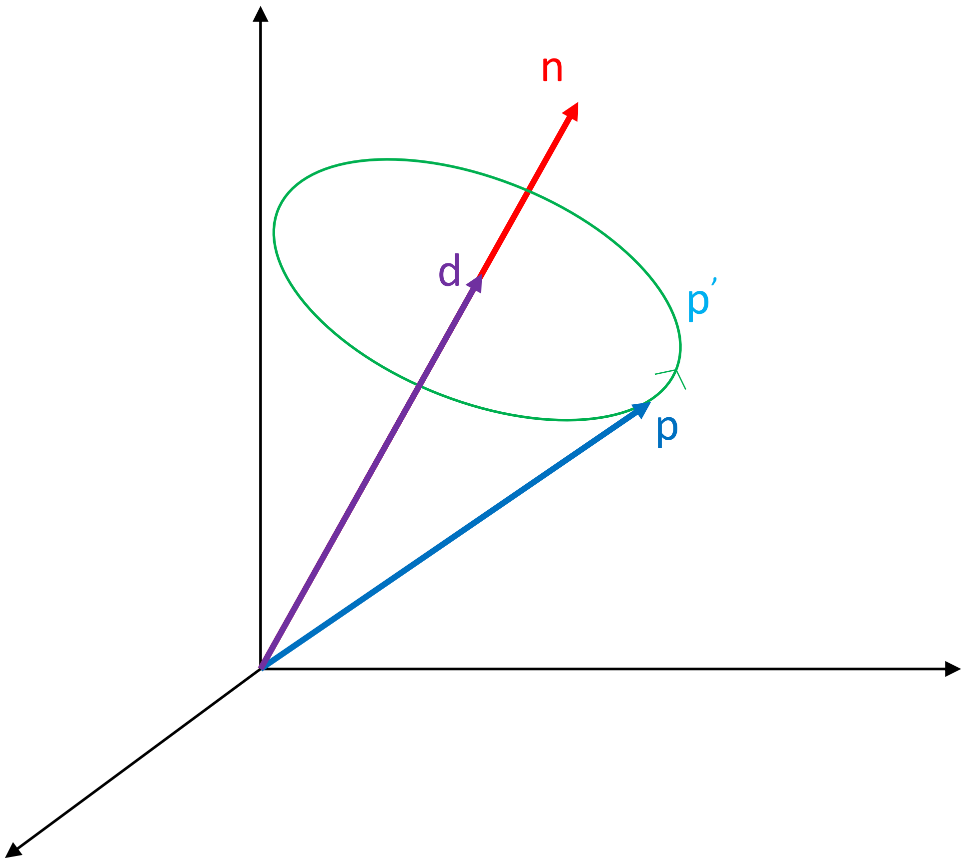

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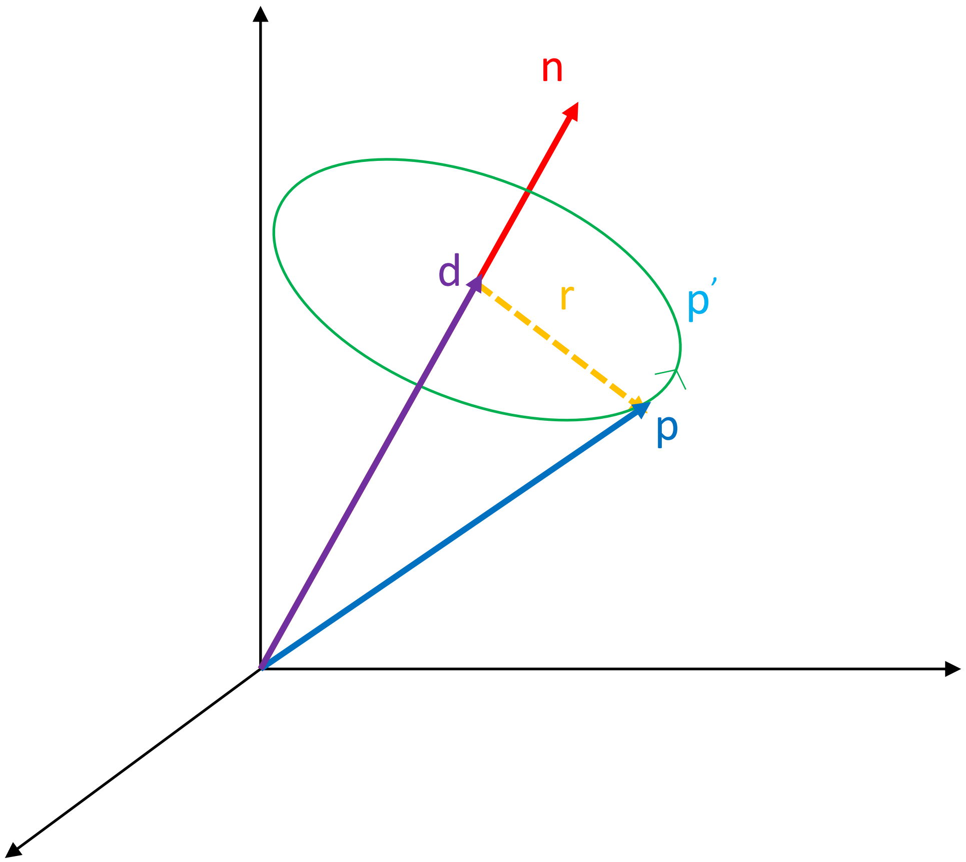

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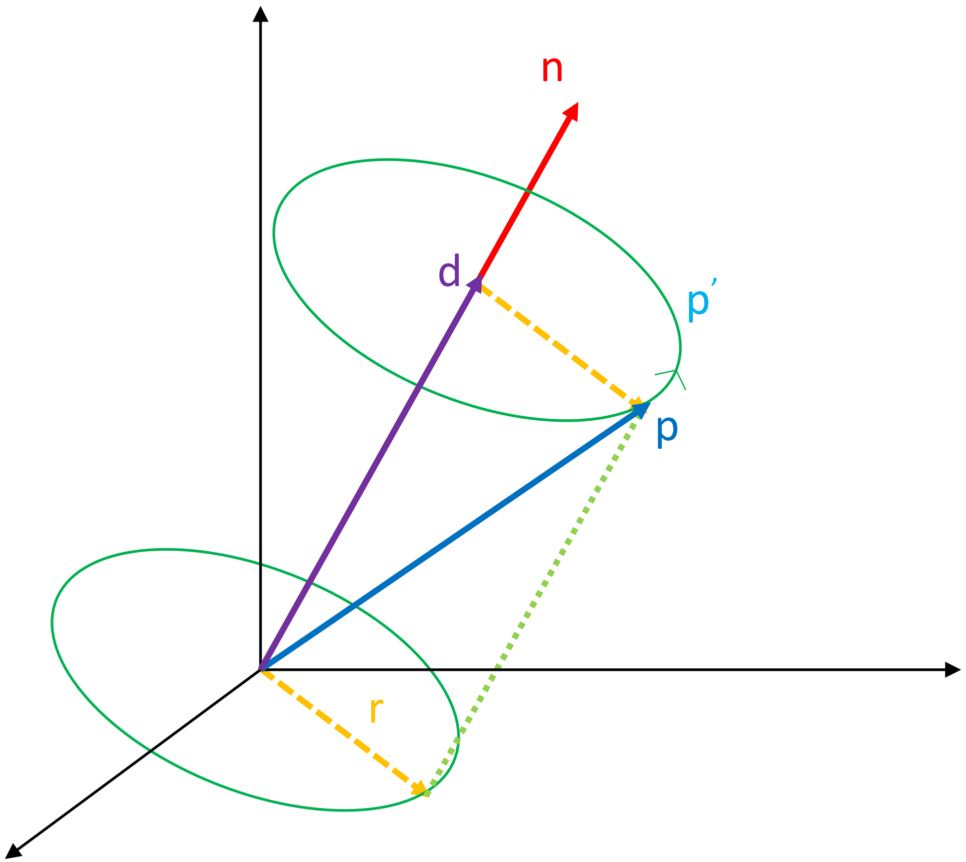

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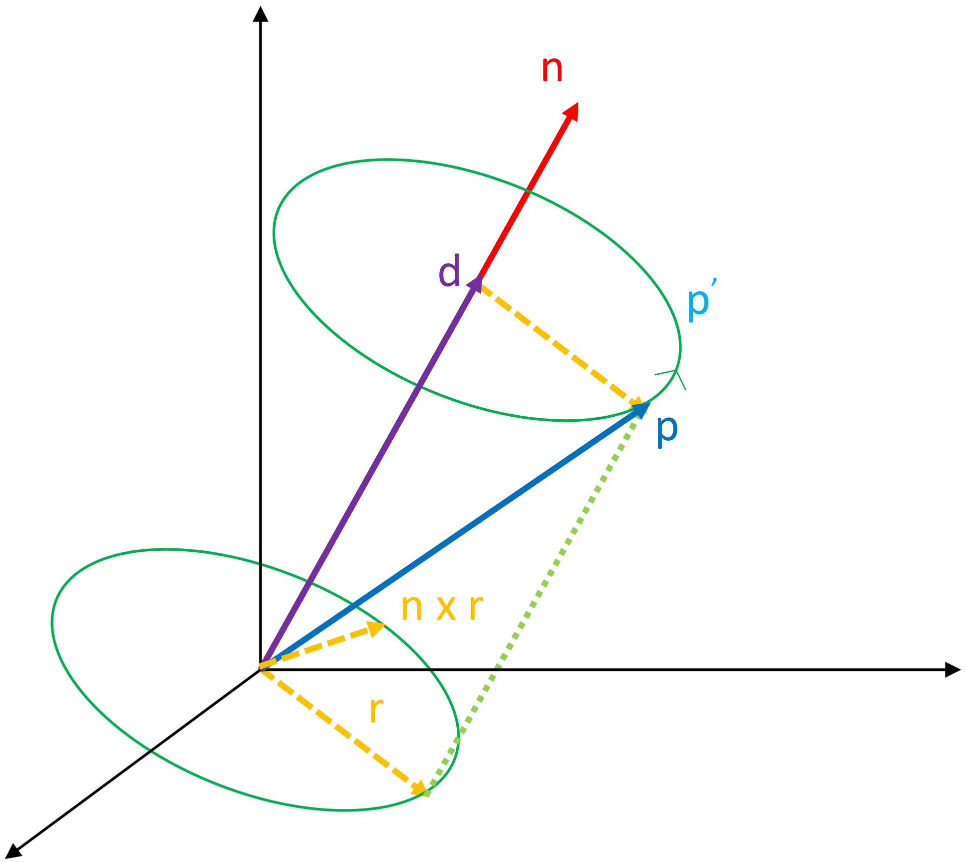

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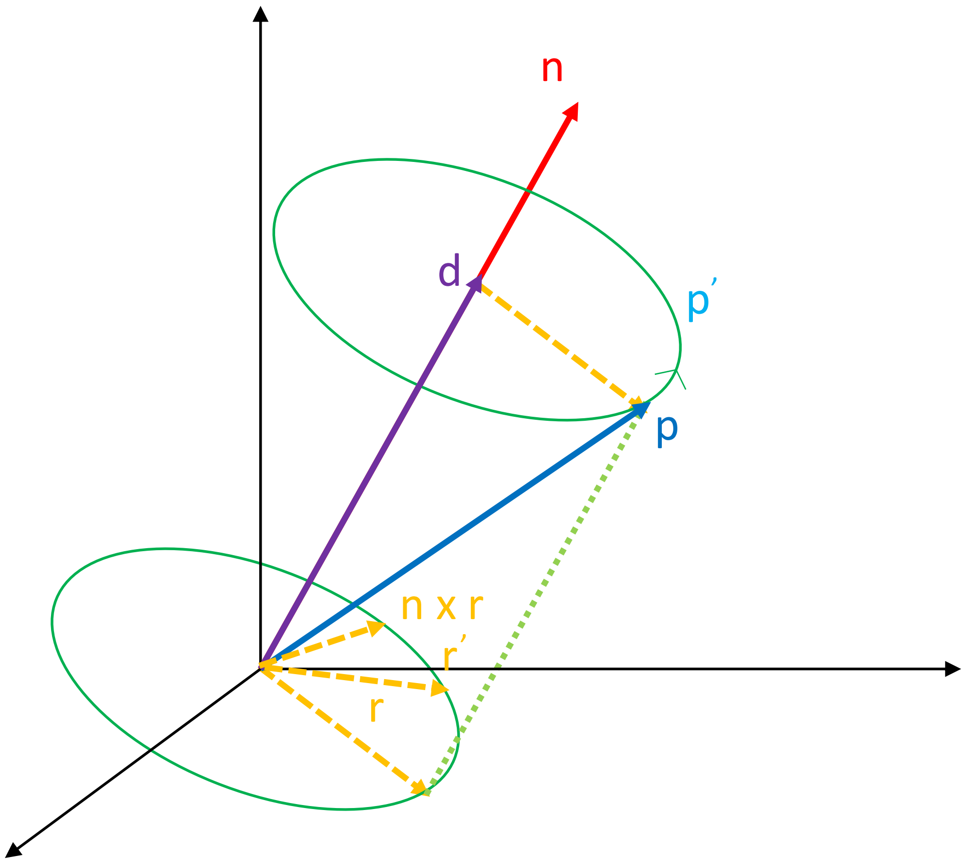

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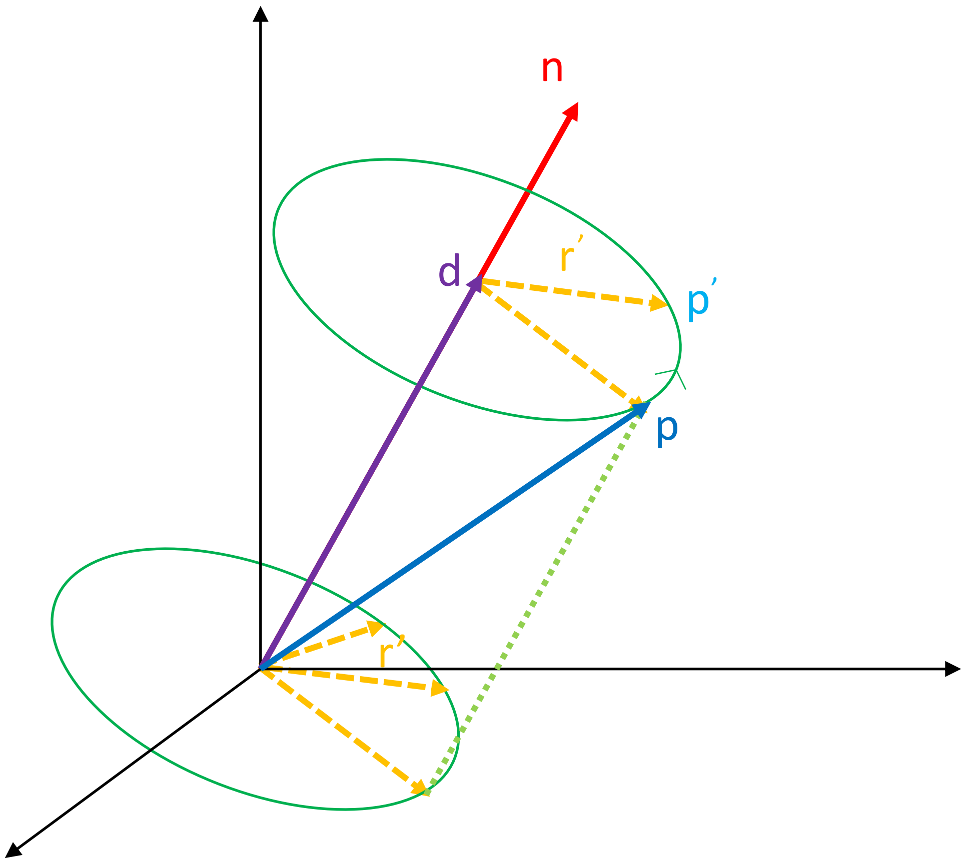

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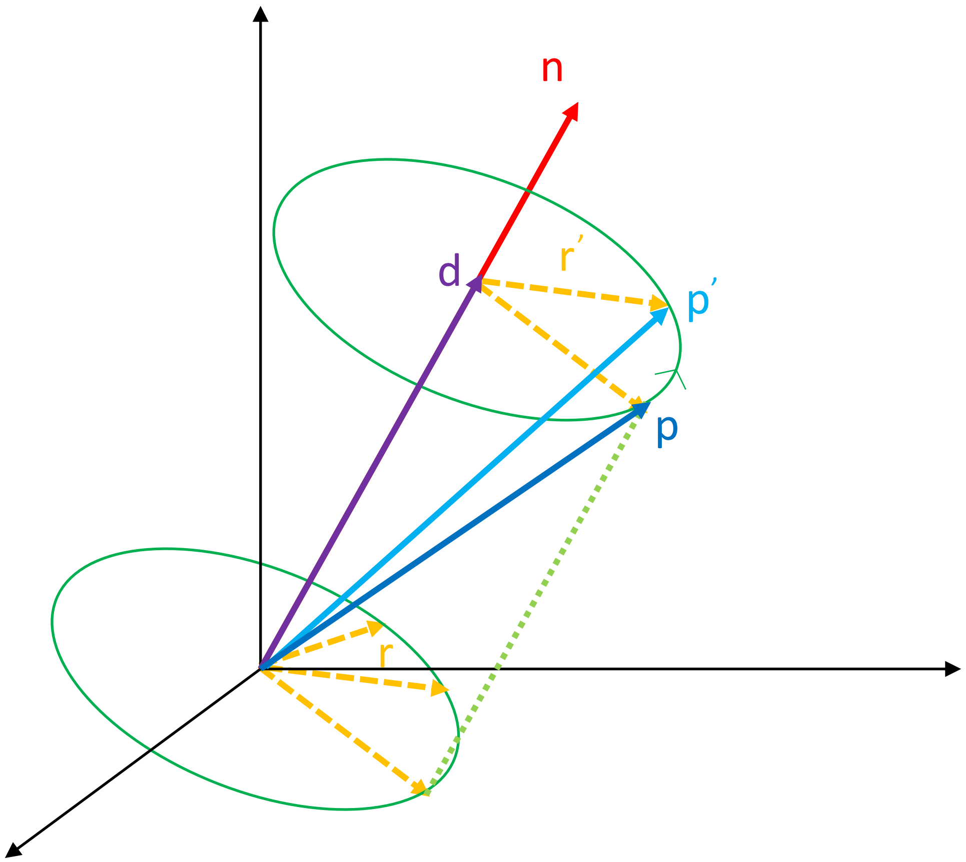

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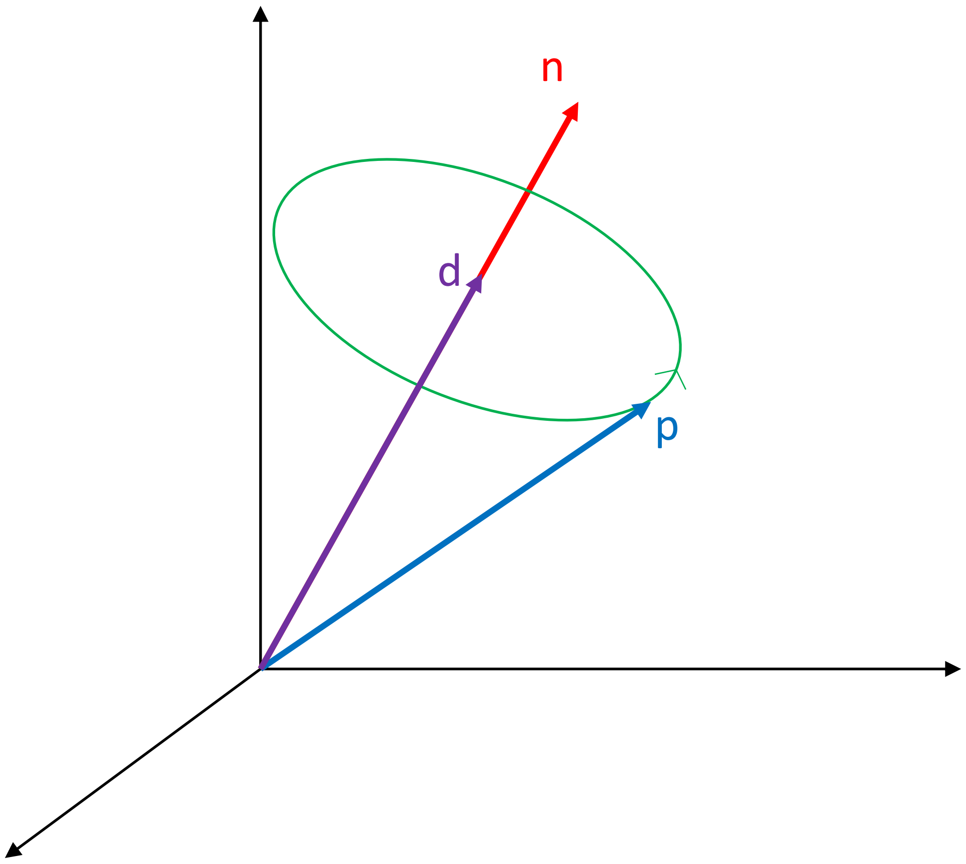
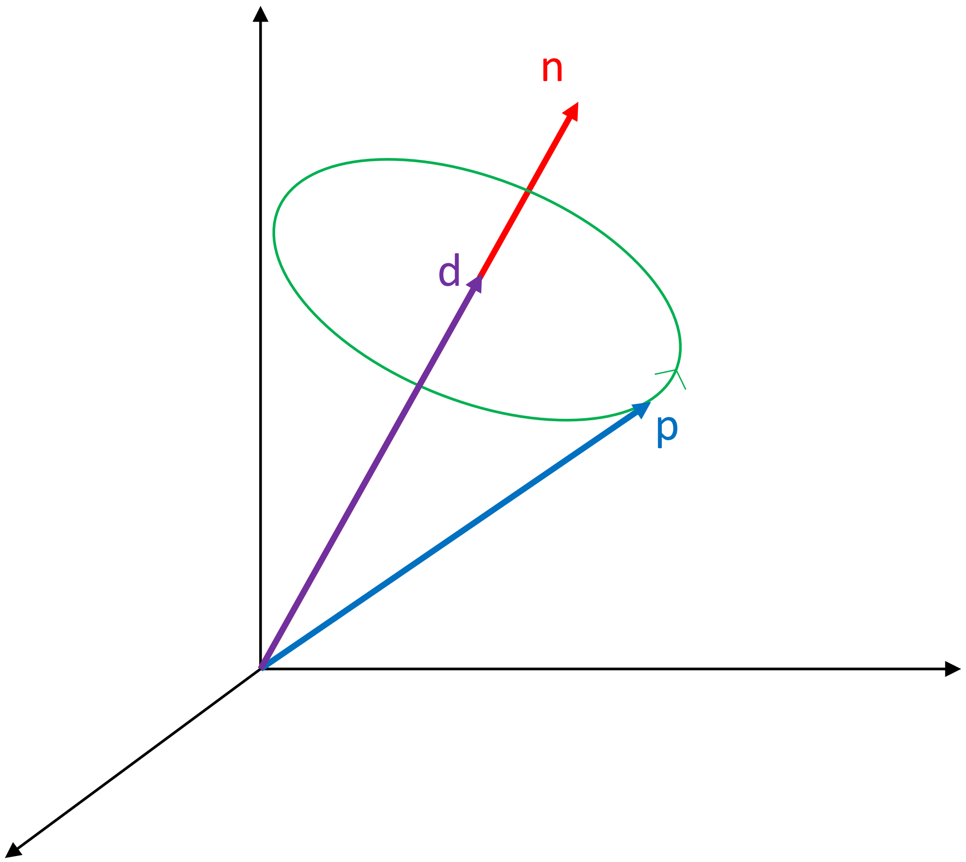


Rodrigues Formula

$$d = (n \cdot p) * n$$



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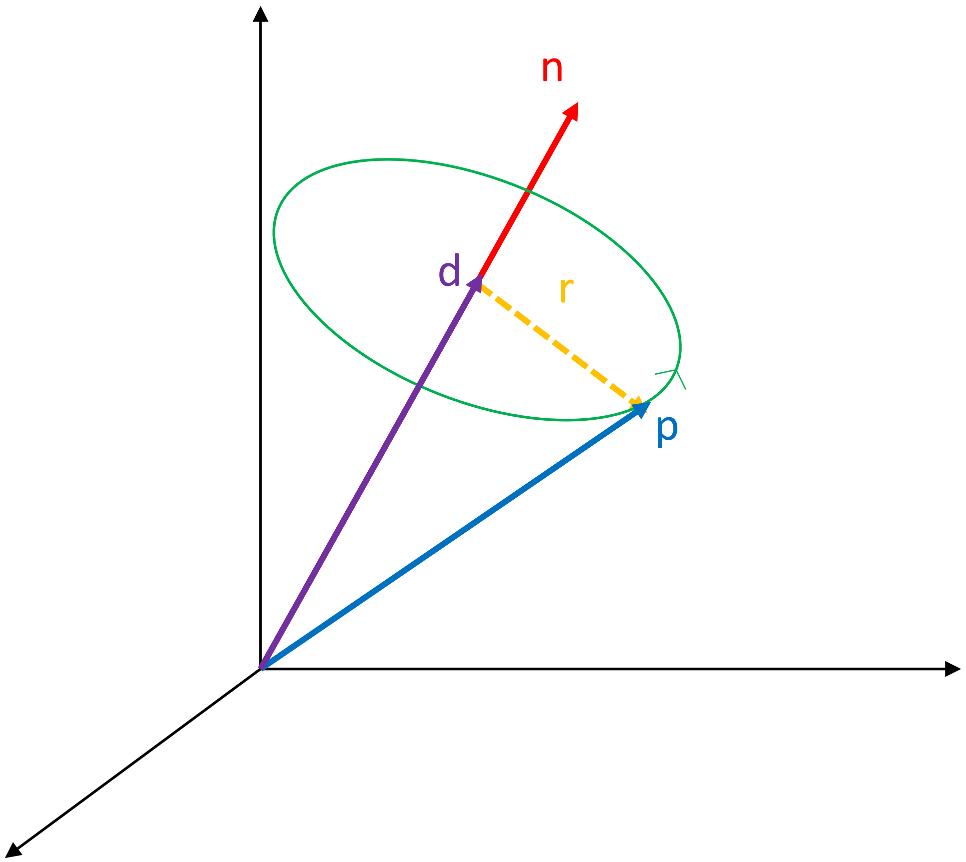

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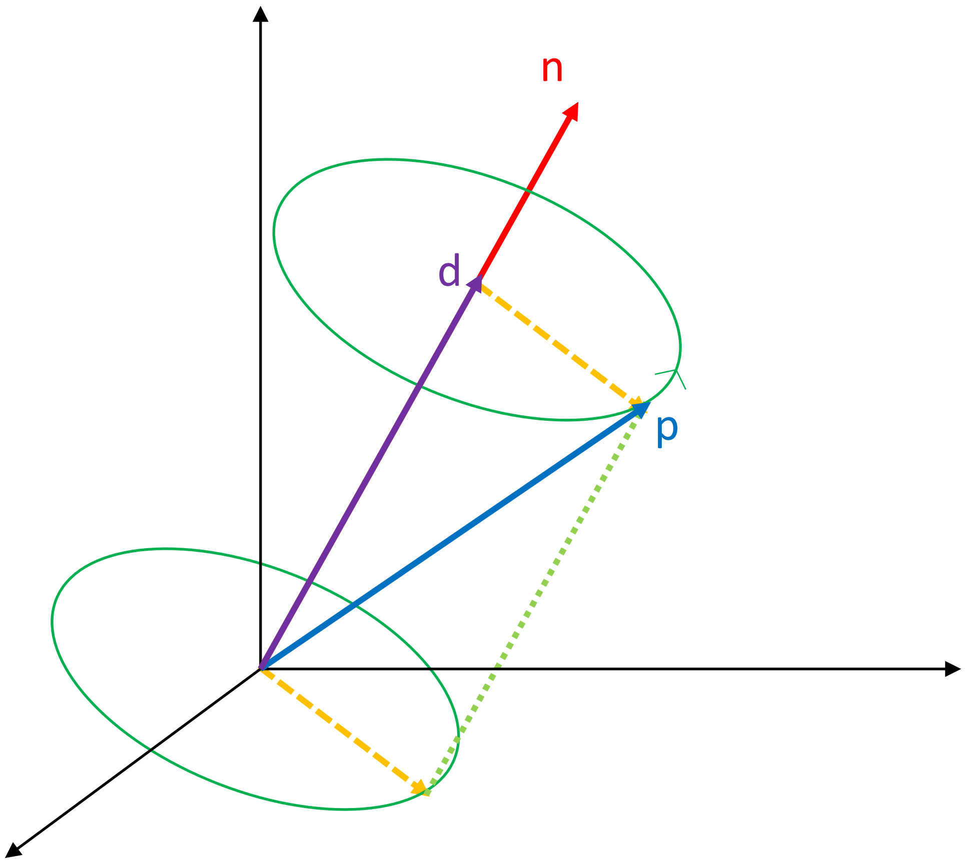

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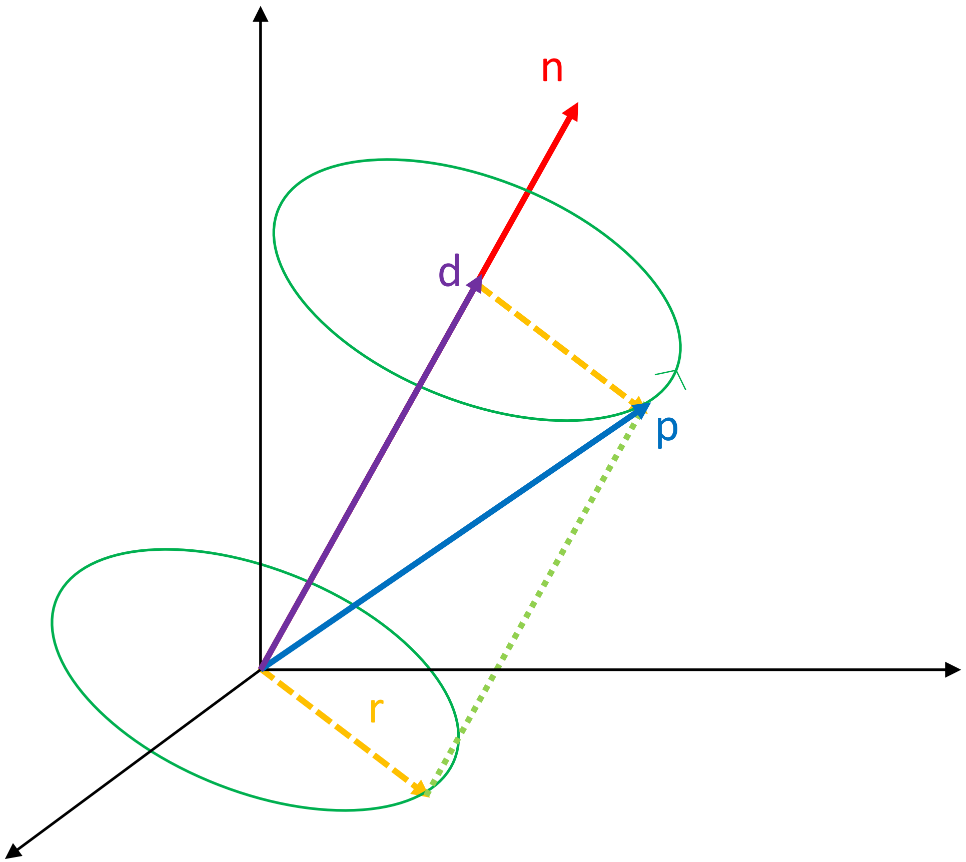
$$d = (n \cdot p) * n$$

$$r = p - d$$



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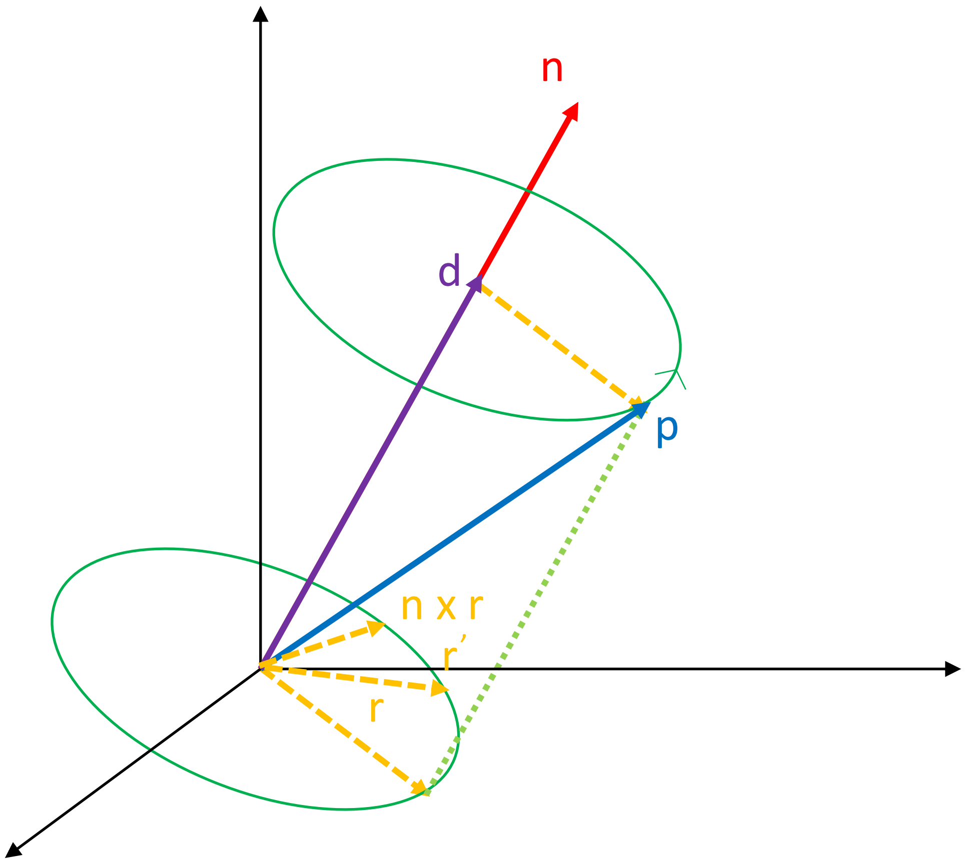

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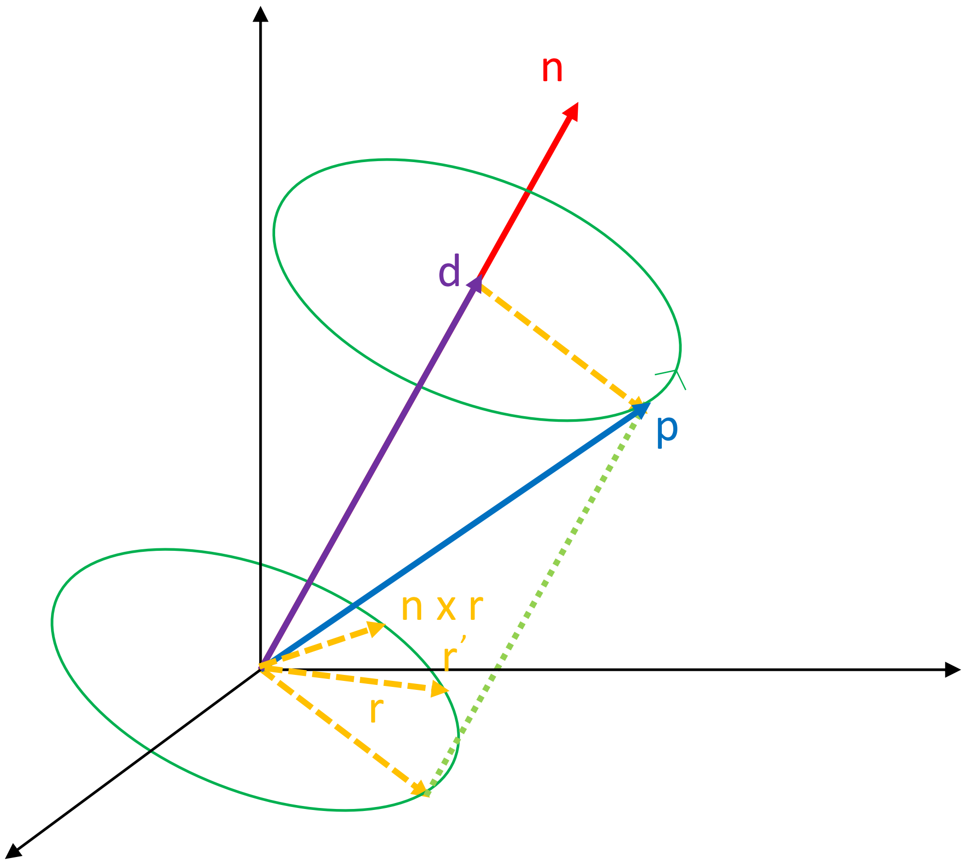
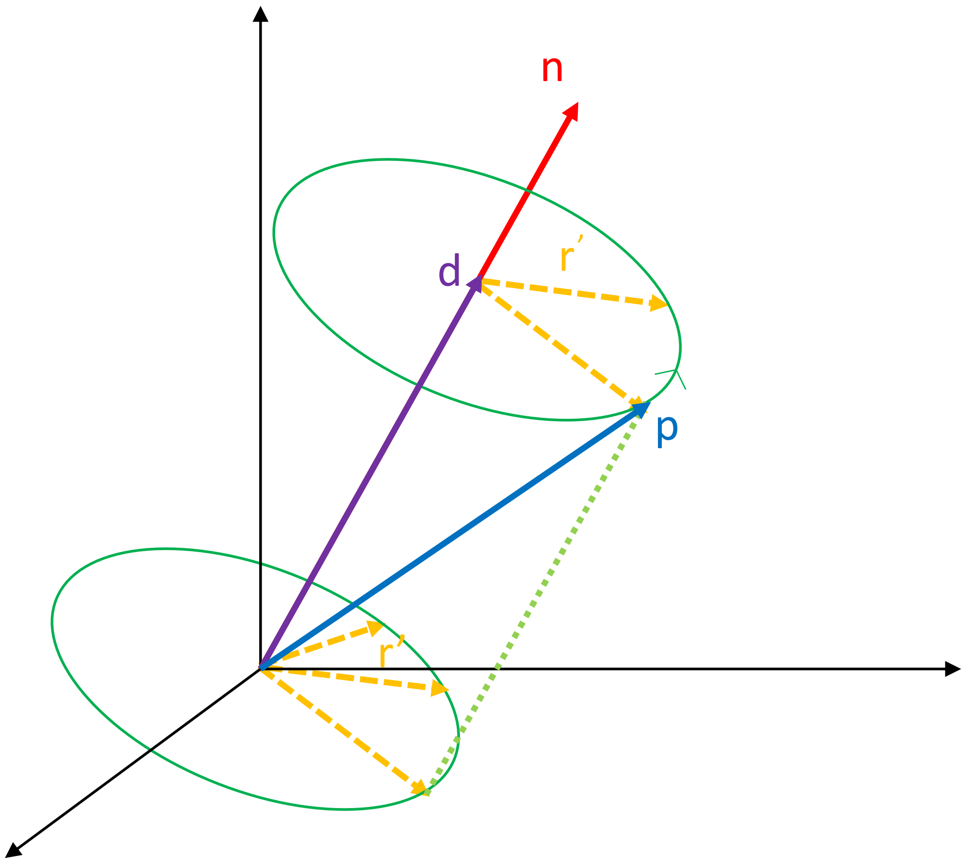


$$d = (n \cdot p) * n$$

$$r = p - d$$

$$r' = r cos\theta + (n \times r) sin\theta$$

Rodrigues Formula



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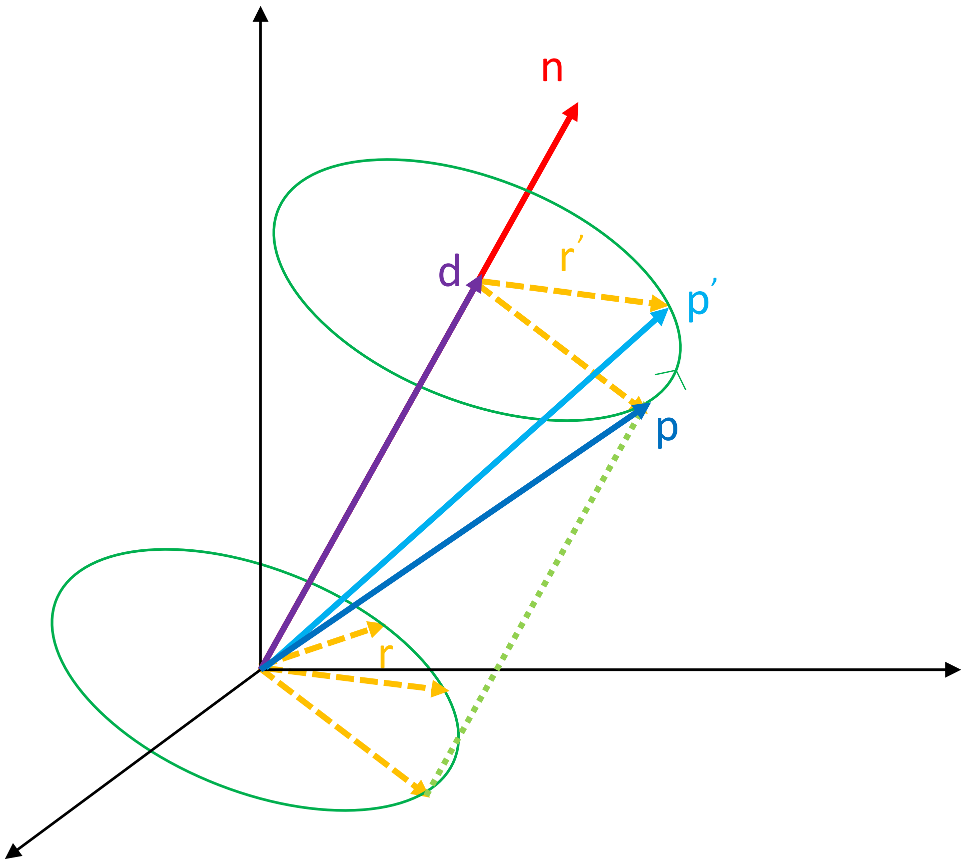

$$d = (n \cdot p) * n$$

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$$\mathrm{p}' = d + r'$$

Euler-Rodrigues Parameters

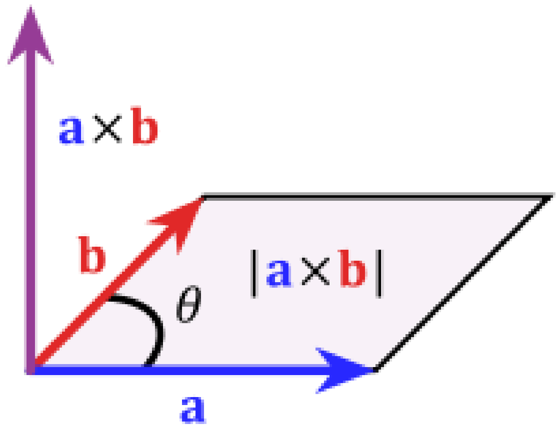
- Given $a^2 + b^2 + c^2 + d^2 = 1$ we can build a rotation
- Set
  - $a = \cos(\theta/2)$
  - $r = (b, c, d) = \sin(\theta/2)n$
- Then rotate using

$$\mathbf{R(a, r, p) = 2a(r \times p) + 2(r \times (r \times p)) + p}$$

(also has a matrix representation)

Axis-Angle Rotation

- Concatenation (yes, with matrices – somewhat difficult)
- Rotation (no Gimbal lock problem)
- Interpolation (find axis of rotation by taking cross product of 2 axes)



Complex Numbers

- 2D numbers: $a + bi$
- Where $i = \sqrt{-1}$

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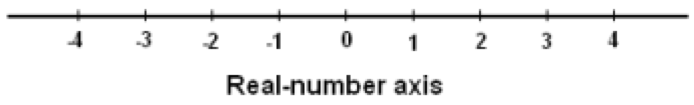
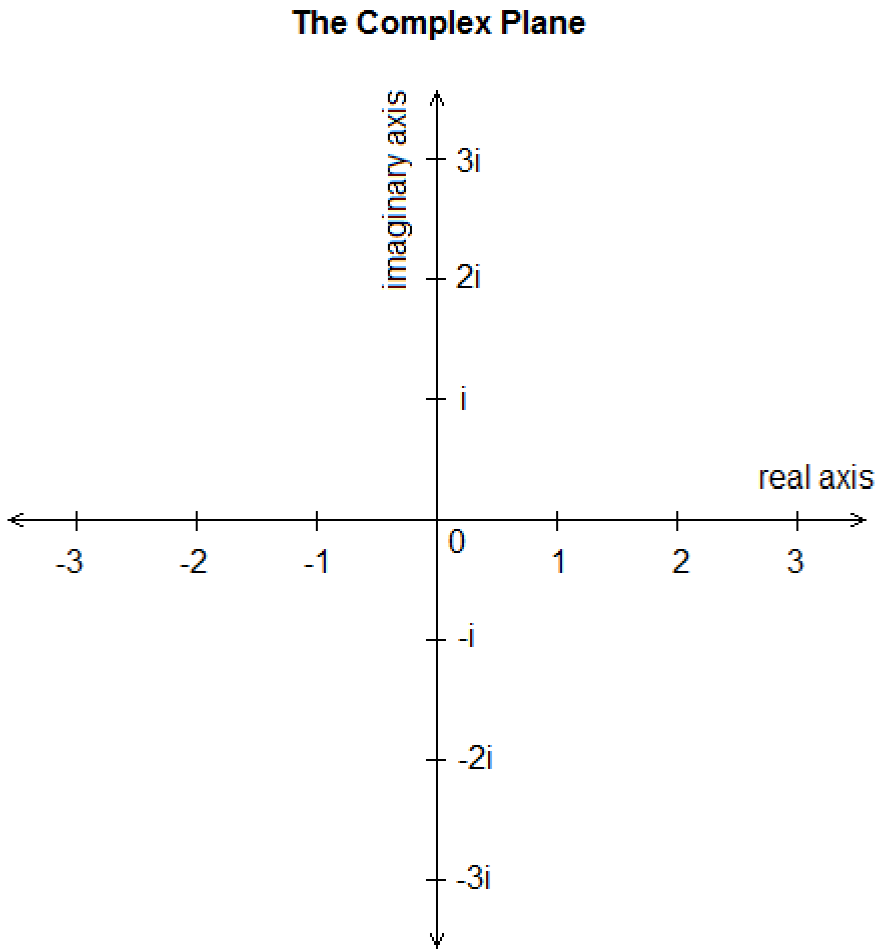
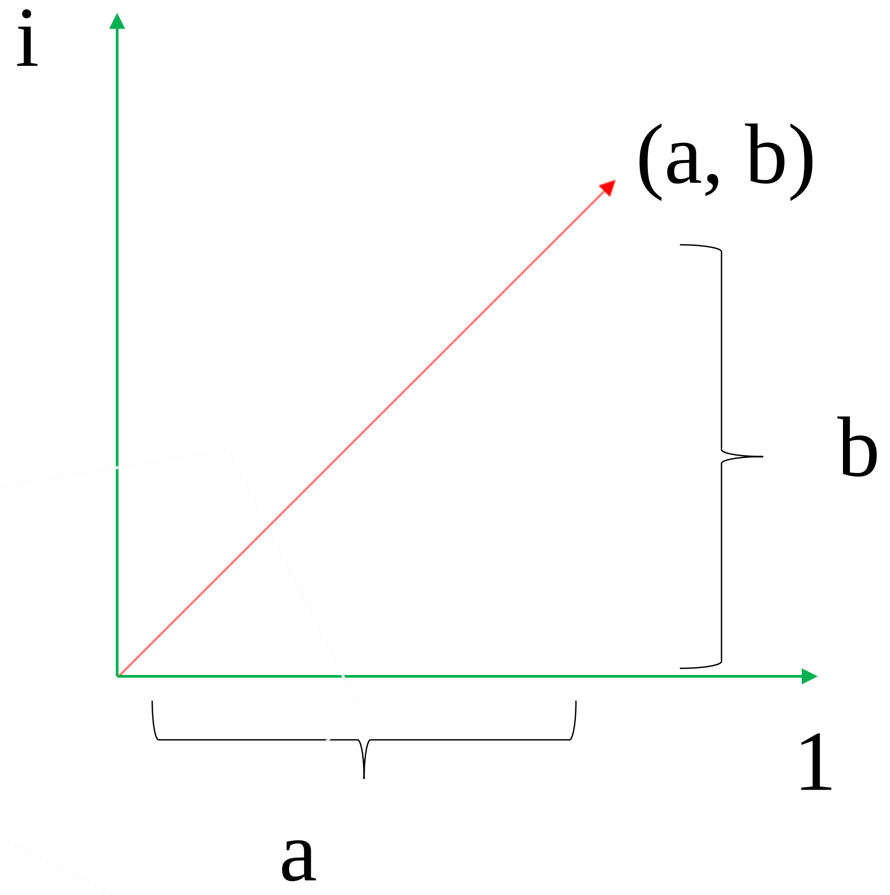




Complex Numbers



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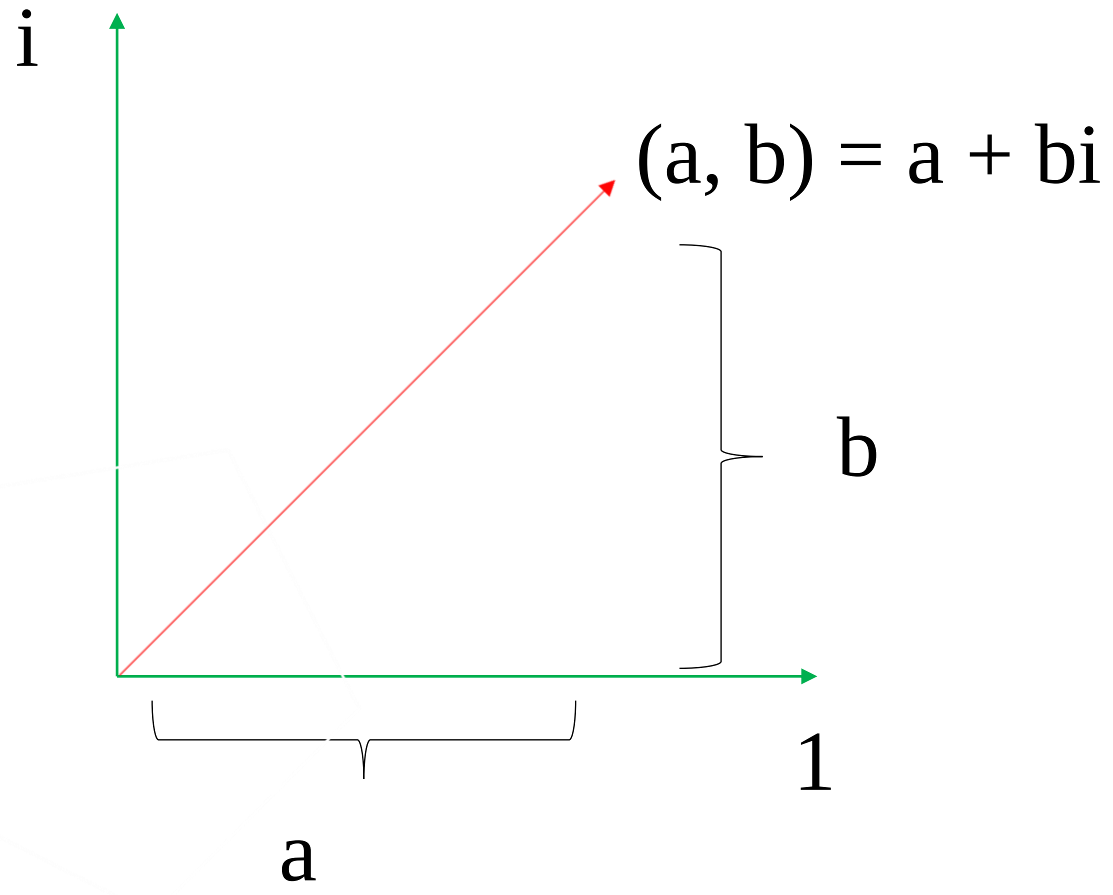
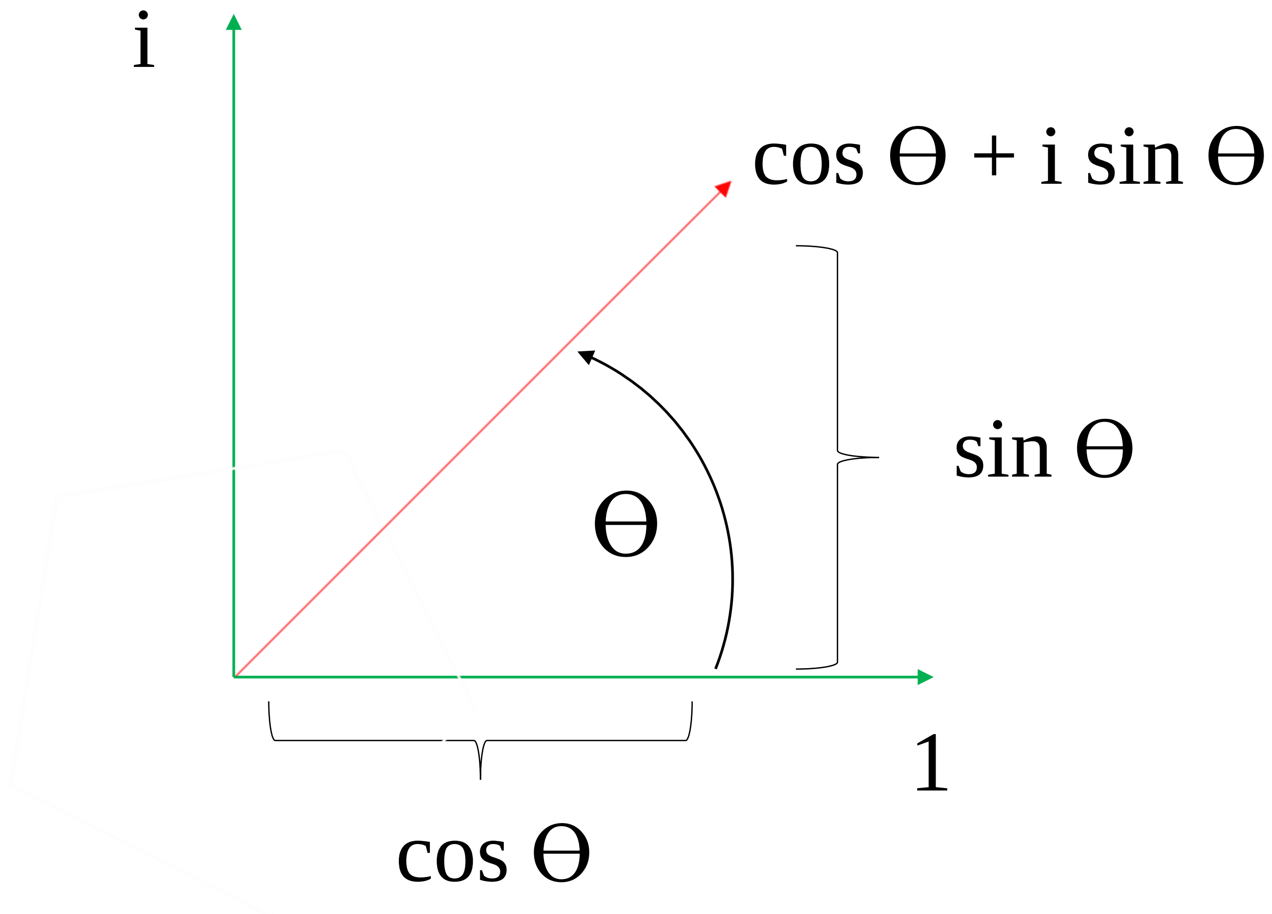


Complex Numbers

- Multiplication:
- $(a + bi)(c + di) = (ac - bd) + (bc + ad)i$

- $(a, b)(c, d) = (ac - bd, bc + ad)$

Complex Numbers: Unit Length



Complex Numbers

- Multiply general complex number by unit complex number:

- $(x + yi)(cos\theta + isin\theta) = (xcos\theta - ysin\theta) + i(xsin\theta + ycos\theta)$

general unit

Complex Numbers

- Multiply general complex number by unit complex number:

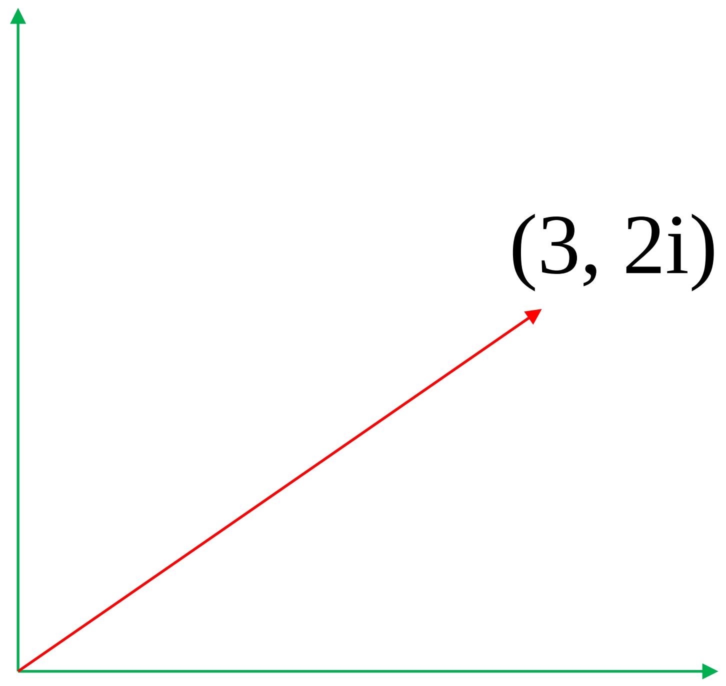
- $(x + yi)(cos\theta + isin\theta) = (xcos\theta - ysin\theta) + i(xsin\theta + ycos\theta)$

general

unit

2D rotation!

Example Complex Number Rotation by i



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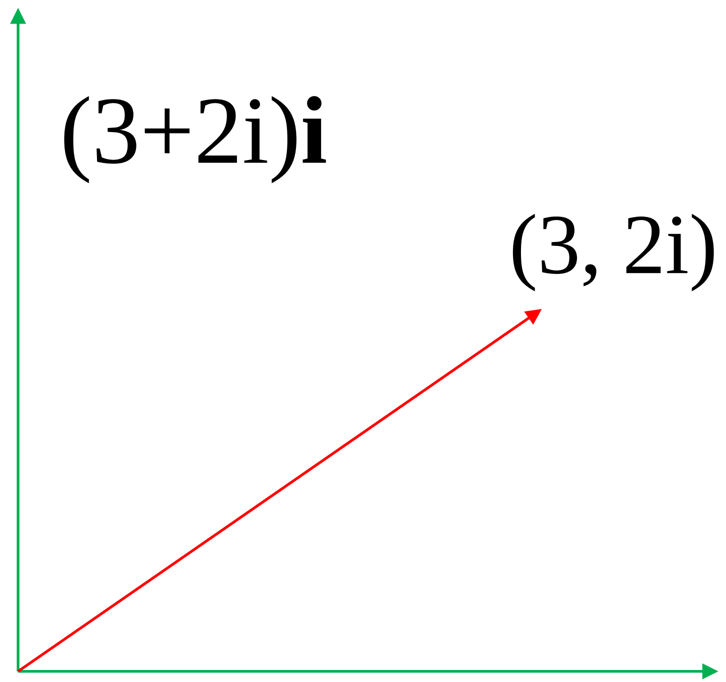
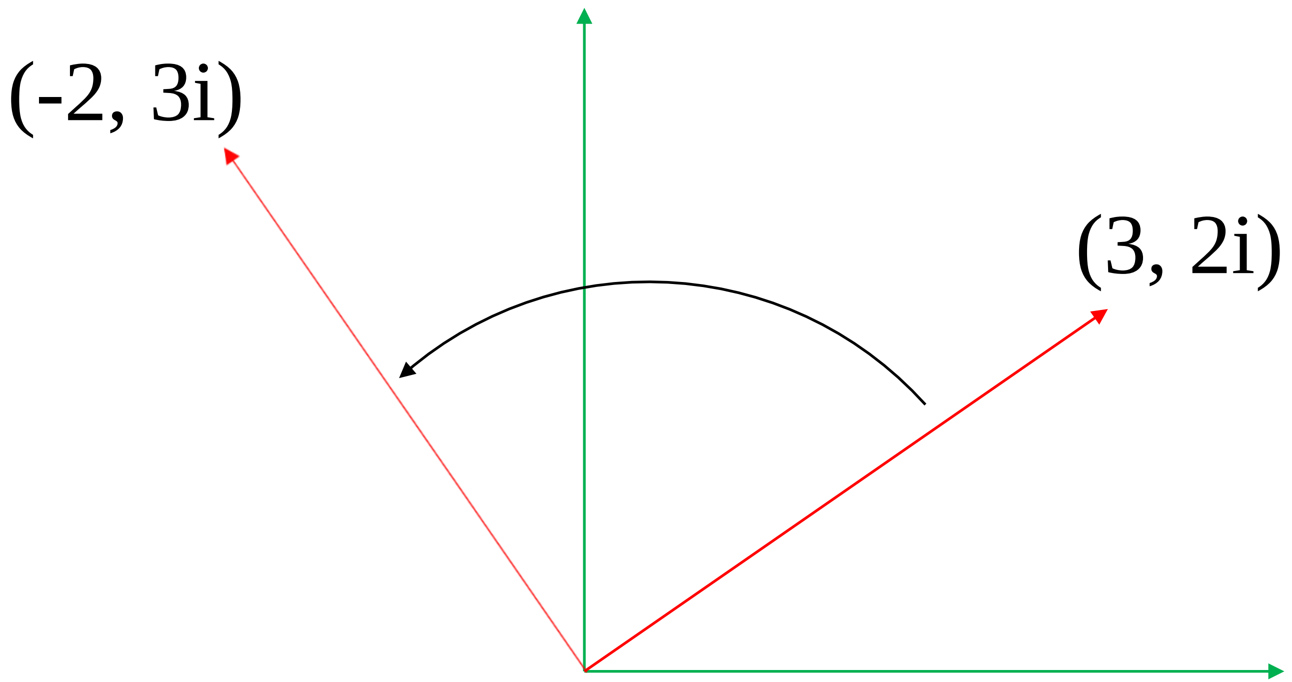
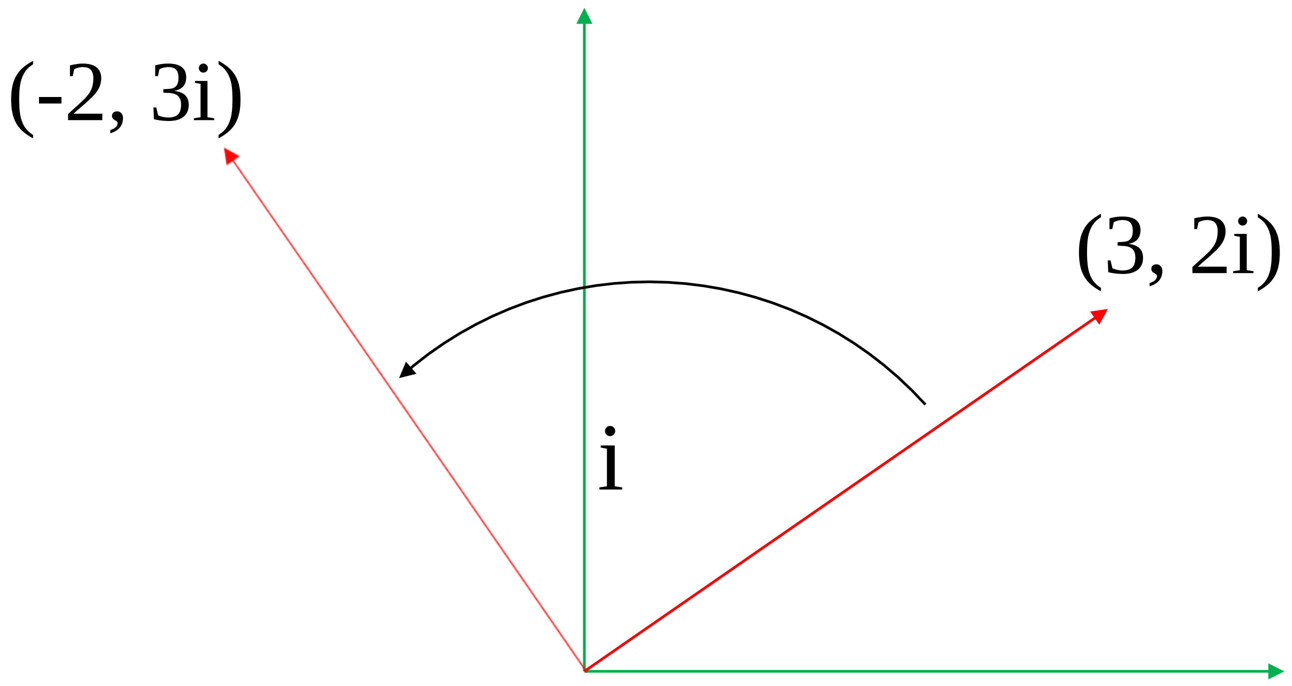


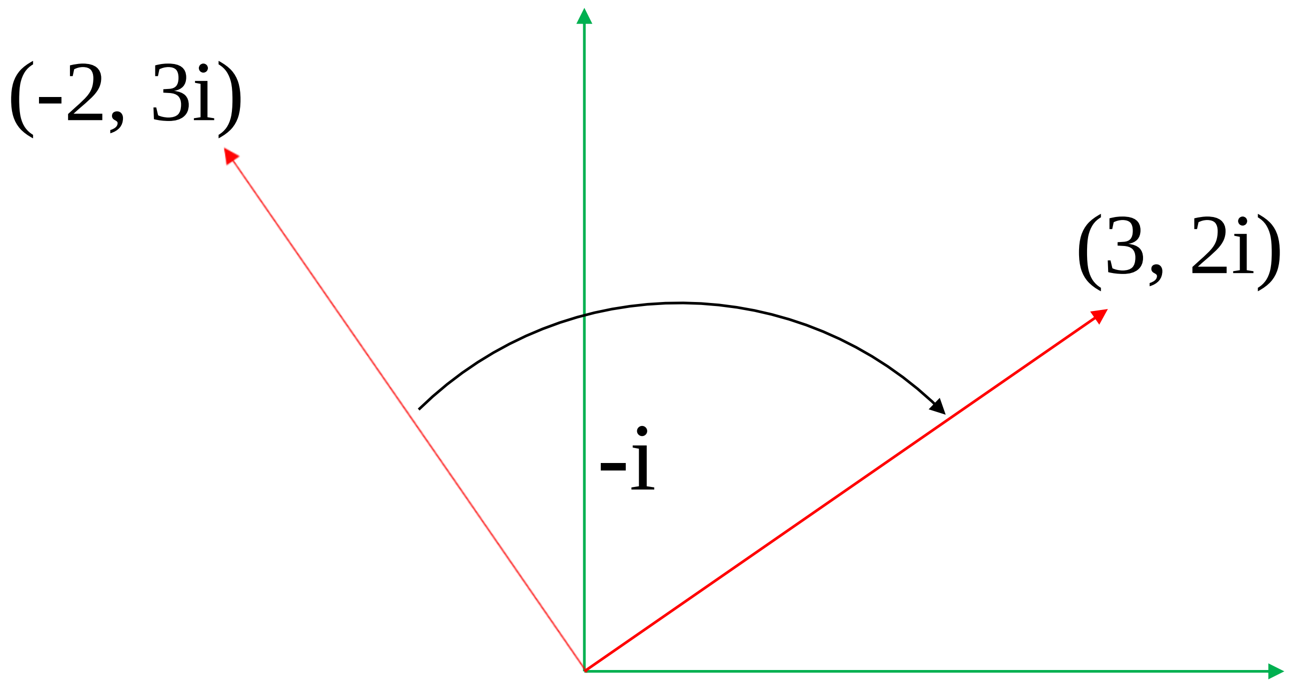
Example Complex Number Rotation by i



Unit Complex Numbers are Rotations!



Inverse Rotation



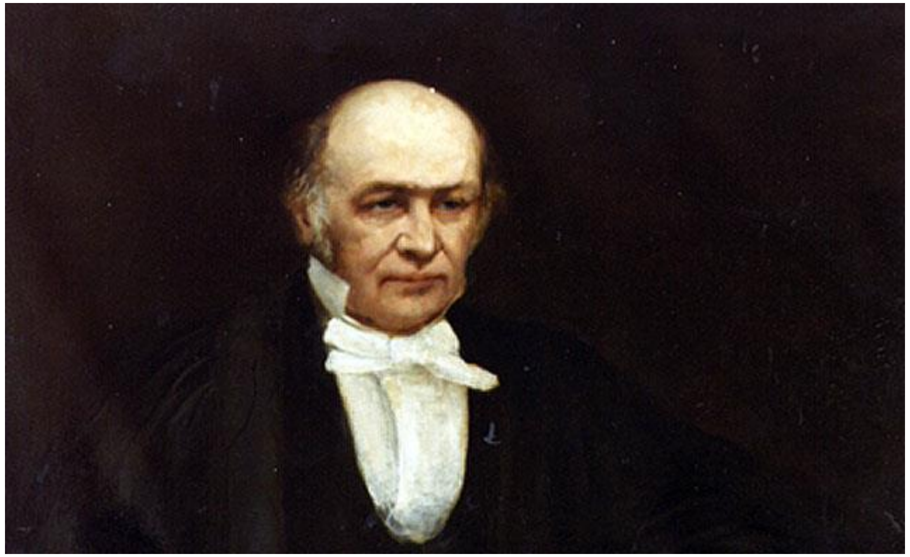
Computational Efficiency?

- Rotation matrix vs. complex number multiplication

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} \qquad (p_x, p_y)(u_x, u_y)$$

3D Complex Numbers?

$$(a_0 + b_0 i + c_0 j)$$



3D Complex Numbers?

- Problem with multiplication

$$(a_0 + b_0 i + c_0 j)(a_1 + b_1 i + c_1 j)$$

3D Complex Numbers?

- Problem with multiplication

$$
\begin{aligned}
(a_0 + b_0 i + c_0 j)(a_1 + b_1 i + c_1 j) = \quad & (a_0 a_1 - b_0 b_1 - c_0 c_1) \\
+ \quad & (a_0 b_1 + b_0 a_1) i \\
+ \quad & (a_0 c_1 + c_0 a_1) j \\
+ \quad & b_0 c_1 ij + c_0 b_1 ji
\end{aligned}
$$

3D Complex Numbers?

- Problem with multiplication

$$
\begin{aligned}
(a_0 + b_0 i + c_0 j)(a_1 + b_1 i + c_1 j) = \quad & (a_0 a_1 - b_0 b_1 - c_0 c_1) \\
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+ \quad & b_0 c_1 ij + c_0 b_1 ji
\end{aligned}
$$

Not a division algebra → not every non-zero number has an inverse!

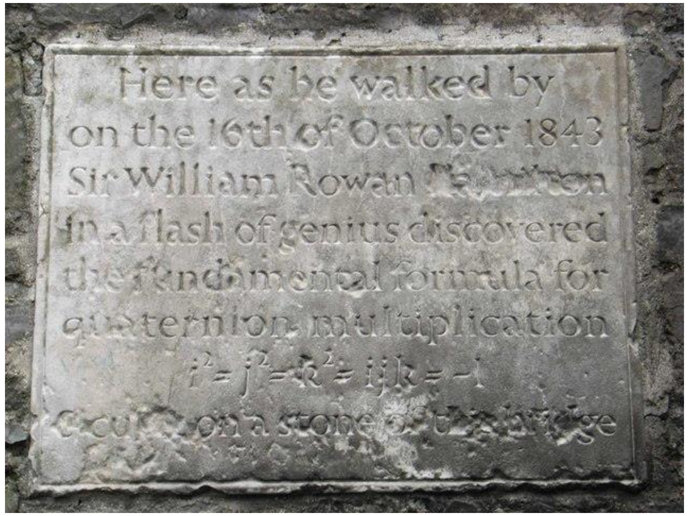
Quaternions

- Hamilton's insight: multiplication possible with 3 imaginary values

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k \qquad jk = i \qquad ki = j$$

$$ji = -k \qquad kj = -i \qquad ik = -j$$



Quaternions

- Are extended complex numbers in 4D

$$a + bi \rightarrow w + xi + yj + zk$$

$$(w, x, y, z)$$

$$(w, v)$$

Quaternions extend Complex Numbers

- Suppose y = z = 0 then,

$$a + bi \rightarrow w + xi + 0j + 0k = w + xi$$

Quaternion Multiplication

- Provides concatenation of rotations
- Take $\mathbf{q}_0 = (w_0, \mathbf{v}_0)$ $\mathbf{q}_1 = (w_1, \mathbf{v}_1)$

$$\mathbf{q}_1\mathbf{q}_0 = (w_1 w_0 - \mathbf{v}_1 \bullet \mathbf{v}_0, w_1\mathbf{v}_0 + w_0\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_0)$$

- If w_0, w_1 are zero:

$$\mathbf{q}_1\mathbf{q}_0 = (-\mathbf{v}_1 \bullet \mathbf{v}_0, \mathbf{v}_1 \times \mathbf{v}_0)$$

- Non-commutative:

$$\mathbf{q}_1\mathbf{q}_0 \neq \mathbf{q}_0\mathbf{q}_1$$

Quaternions

- Identity quaternion is $(1, 0, 0, 0)$
  - applies no rotation
  - remains at reference orientation
- $\mathbf{q}^{-1}$ is inverse
  - $\mathbf{q} \cdot \mathbf{q}^{-1}$ gives identity quaternion

Quaternion Rotation

- Like complex numbers, unit quaternions represent rotations
- For 3D rotation:

$$w = \cos(\theta/2)$$

$$(x, y, z) = v = \sin(\theta/2)n$$

Quaternion Rotation

- We can easily obtain the **inverse quaternion** for any unit (normalized) quaternion

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$$q^{-1} = (w, v)^{-1} = (\cos(-\theta/2), \sin(-\theta/2)n)$$

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$$(w, v)^{-1} = (w, -v)$$

Quaternion Rotation

- Have vector **p**, unit quaternion *q*
- Treat **p** as quaternion *p* = (0, **p**)
- 3D rotation of *p* by *q* is

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Quaternion Rotation

- **Quaternion Rotation**
- **Euler-Rodrigues**

- $w = \cos(\theta/2)$
- $a = \cos(\theta/2)$

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- **Euler-Rodrigues**

- $a = \cos(\theta/2)$
- $(b, c, d) = r = \sin(\theta/2)n$

Quaternion Rotation

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- $(b, c, d) = r = \sin(\theta/2)n$

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Matrix Form

- Decompose 2D rotation matrix into:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Set

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Then

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta\mathbf{I} + \sin\theta\mathbf{J}$$

Matrix Form

- **I** and **J** follow same rules as 1 and i

$$\mathbf{J}^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$$

$$(a\mathbf{I} + b\mathbf{J})(c\mathbf{I} + d\mathbf{J}) = (ac - bd)\mathbf{I} + (bc + ad)\mathbf{J}$$

- Complex numbers in another form!

Matrix Form

- For quaternions:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Matrix Form

- Then

$$X^2 = Y^2 = Z^2 = XYZ = -I$$

- And

$$Q = wI + xX + yY + zZ$$

- Q is a quaternion!

Matrix Form

$$\mathbf{Q} = \begin{bmatrix} w & -z & y & x \\ z & w & -x & y \\ -y & x & w & z \\ -x & -y & -z & w \end{bmatrix}$$

- If $w^2 + x^2 + y^2 + z^2 = 1$
  - Unit vectors
  - Orthogonal vectors, dot products 0
  - Determinant 1
- Q is a rotation matrix!

$$\mathbf{P}' = \mathbf{QPQ}^{-1}$$

Matrix Form

- $\mathbf{Q}$ is a rotation quaternion
- Transform with half angle

$$\mathbf{P}' = \mathbf{Q}\mathbf{P}\mathbf{Q}^{-1}$$

Rotate halfway   Rotate halfway

4D Rotation

- Two types:
  - Single rotation (one plane, like 2D/3D)
  - Double rotation (two orthogonal planes)

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90° rotation in (y, z) plane

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$$(w + ix + jy + kz)(i) = -x + iw + jz - ky$$

90° rotation in (w, x) plane

-90° rotation in (y, z) plane

4D Rotation

- Two types:
  - Single rotation (one plane, like 2D/3D)
  - Double rotation (two orthogonal planes)

$$(w + ix + jy + kz)(-i) = x - iw - jz + ky$$

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90° rotation in (y, z) plane

4D Rotation

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  - Single rotation (one plane, like 2D/3D)
  - Double rotation (two orthogonal planes)

$$(i)\ (w + ix + jy + kz)(-i) = 0\ +\ 2(-jz + ky)$$

4D Rotation

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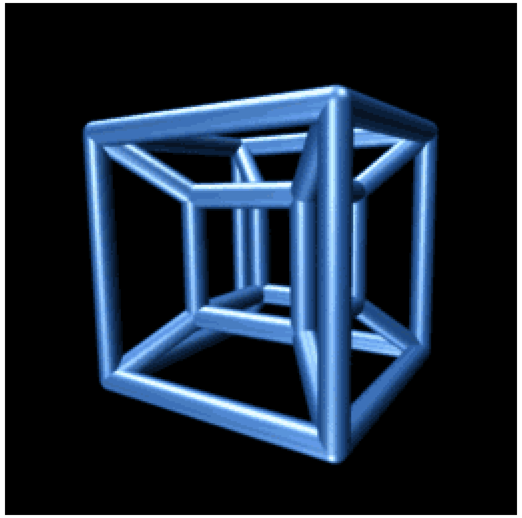

$$(i)\,(w + ix + jy + kz)(-i) = 0\ +\ 2(-jz + ky)$$

4D Rotation

- Two kinds of quaternion rotations (both angles equal) - **isoclinic**
  - Multiply on left, rotate ccw in both planes
  - Multiply on right, rotate ccw in one, cw in other

4D Rotation

- Matched pair of isoclinic rotations

$$\mathbf{P}' = \mathbf{Q}\mathbf{P}\mathbf{Q}^{-1}$$

Rotate by $\theta/2$ ccw in 3D rotation plane and ccw in the orthogonal plane

Rotate by $\theta/2$ ccw in 3D rotation plane and cw in the orthogonal plane

4D Rotation

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Rotate by $\theta/2$ ccw in 3D rotation plane ~~and ccw in the orthogonal plane~~

Rotate by $\theta/2$ ccw in 3D rotation plane ~~and cw in the orthogonal plane~~

Quaternion Rotation

- Can easily concatenate rotations – no trigonometric calculations needed

$$\mathbf{q}_1 \cdot (\mathbf{q}_0 \cdot \mathbf{p} \cdot \mathbf{q}_0^{-1}) \cdot \mathbf{q}_1^{-1} = (\mathbf{q}_1 \cdot \mathbf{q}_0) \cdot \mathbf{p} \cdot (\mathbf{q}_1 \cdot \mathbf{q}_0)^{-1}$$

- Note multiplication order: right-to-left

Creating Quaternions

- So for example, if we want to rotate 90° around z-axis:

$$w = \cos(45^\circ) = \sqrt{2}/2$$

$$x = 0 \cdot \sin(45^\circ) = 0$$

$$y = 0 \cdot \sin(45^\circ) = 0$$

$$z = 1 \cdot \sin(45^\circ) = \sqrt{2}/2$$

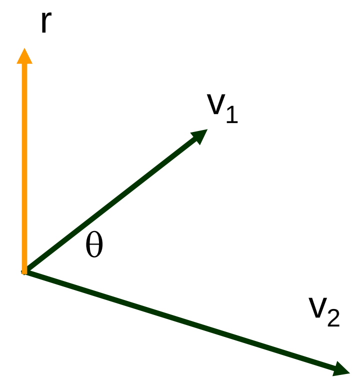
$$\mathbf{q} = (\sqrt{2}/2, 0, 0, \sqrt{2}/2)$$

Creating Quaternions

- Another example
  - We have a vector $\mathbf{v}_1$ and want to rotate to $\mathbf{v}_2$
  - Need rotation axis $\mathbf{r}$, angle θ

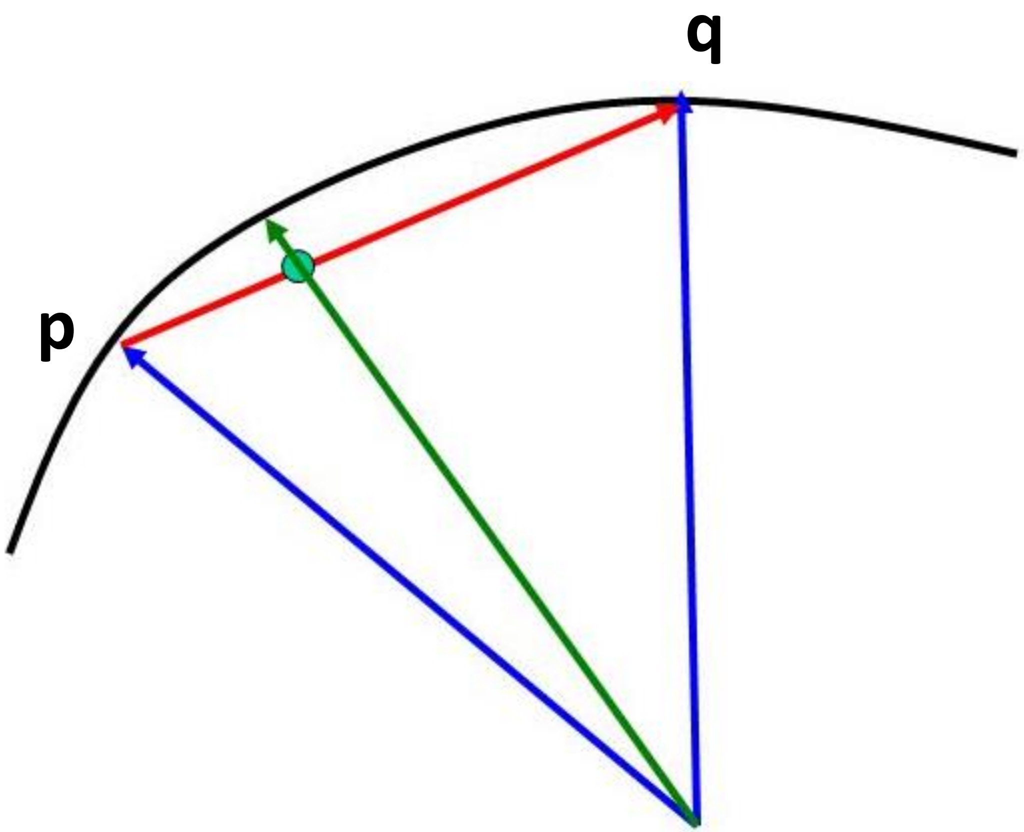
$$\theta = \operatorname{acos}(\hat{\mathbf{v}}_1 \bullet \hat{\mathbf{v}}_2)$$
$$\mathbf{r} = \mathbf{v}_1 \times \mathbf{v}_2$$

  - Plug into previous formula



Interpolating quaternions

- E.g. with linear interpolation: $(1-t)\mathbf{p} + t\mathbf{q}$



Summary

- 2D rotations → unit complex numbers
- 3D rotations → unit quaternions

- Why four values → division algebra
- i, j, k are imaginary axes
- qpq^{-1} is a two step rotation because we can't rotate in a single plane using quaternions in a single step
- Rotation, concatenation and interpolation

glm::gtc::quaternion Namespace Reference

GLM_GTC_quaternion extension: Quaternion types and functions. More...

Typedefs

typedef detail::tquat< double >	**dquat** Quaternion of double-precision floating-point numbers.
typedef detail::tquat< float >	**fquat** Quaternion of single-precision floating-point numbers.
typedef detail::tquat< detail::thalf >	**hquat** Quaternion of half-precision floating-point numbers.
typedef detail::tquat< float >	**quat** Quaternion of floating-point numbers.

Functions

template<typename T > detail::tquat< T >	**conjugate** (detail::tquat< T > const &q) Returns the q conjugate.
template<typename T > detail::tquat< T >	**cross** (detail::tquat< T > const &q1, detail::tquat< T > const &q2) Returns the cross product of q1 and q2.
template<typename T > detail::tquat< T >::value_type	**dot** (detail::tquat< T > const &q1, detail::tquat< T > const &q2) Returns dot product of q1 and q2, i.e., q1[0] * q2[0] + q1[1] * q2[1] + ...
template<typename T > detail::tquat< T >	**inverse** (detail::tquat< T > const &q) Returns the q inverse.