GRK 9

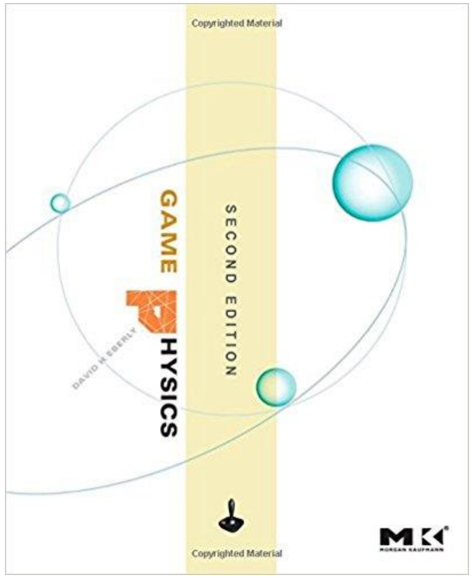
Dr Wojciech Palubicki

Next Lectures

- **Physics**: translational and rotational Newtonian physics, equations of motion for a system of bodies (**rigid body dynamics**)
- **Numerical methods** for integration
- **Collision detection** and Spatial Structures
- **Collision response**

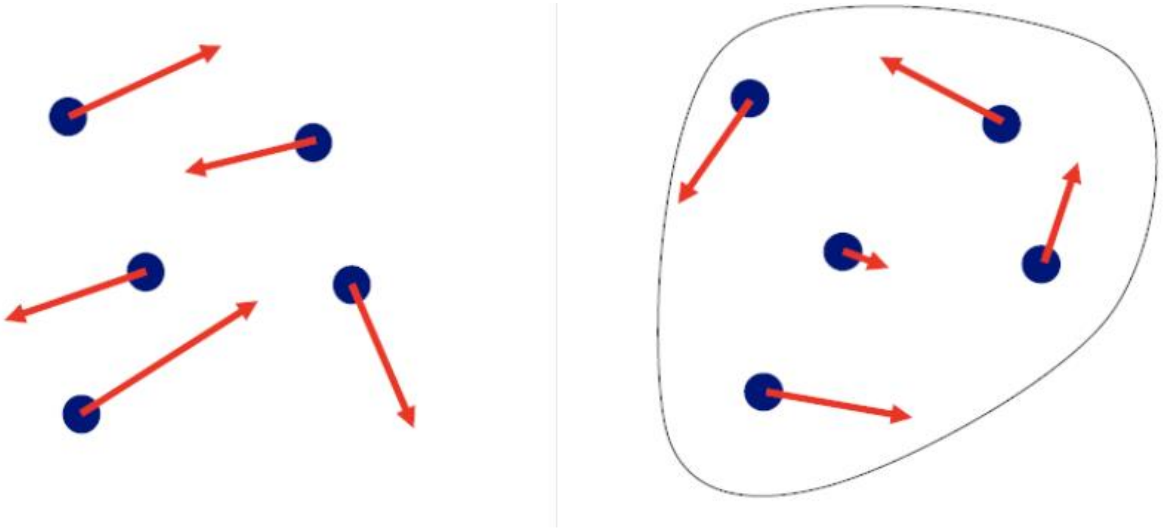
References

- Game Physics, D. Eberly
- Real-Time Collision Detection, Christer Ericson



Rigid body

- A **particle system** is a collection of a finite number of particles p_i with defined positions and masses (and possibly other attributes).
- A **rigid body** is defined by the region that its mass is at (can be thought of as a particle system inside a region where relative distances between particles don't change).



particle system **rigid body**

Physical model for a rigid body

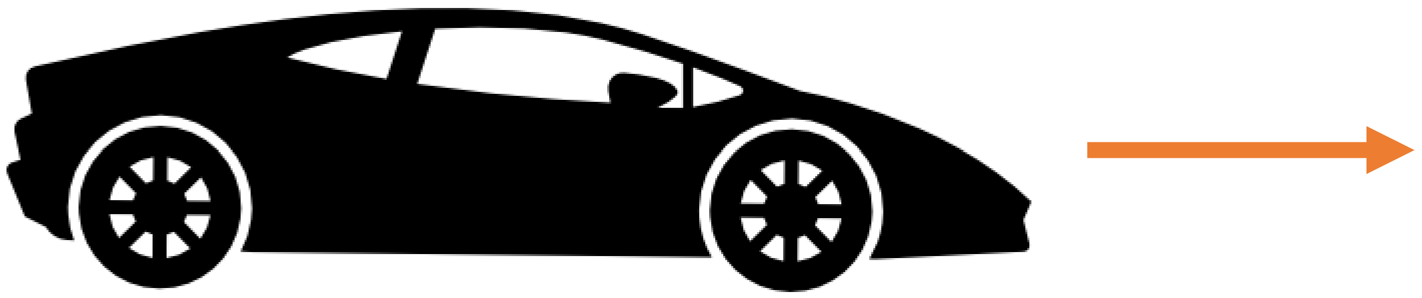
- Describe motion
- Position, Velocity and Acceleration
- Rotation, Angular velocity and Torque
- Cartesian coordinates in 2D and 3D

Newton’s laws of motion

- Linear momentum
- Angular momentum

Newton’s laws of motion

- <u>Linear momentum</u>
- Angular momentum



Newton’s laws of motion

- Linear momentum
- <u>Angular momentum</u>

Angular momentum

- Center of mass
- Inertia
- Torque

Particle kinematics 2D

- A particle moving across the xy plane
- Position at time t is

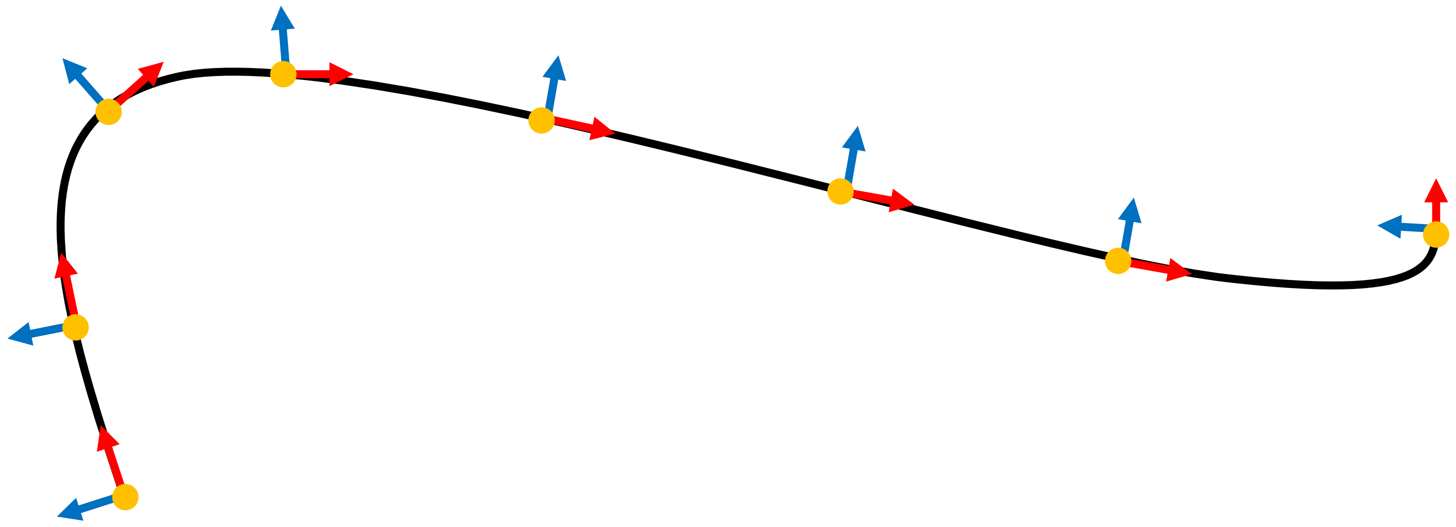
$$r(t) = x(t) + y(t)$$

or $(x(t), y(t))$

Particle kinematics 2D

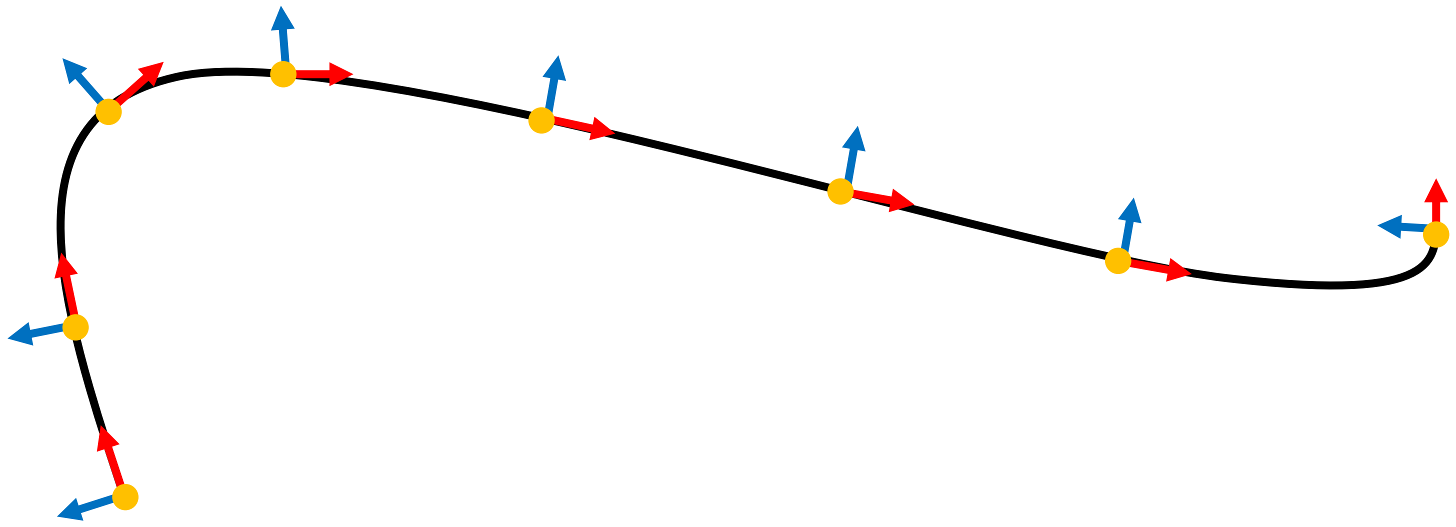
- Velocity at time t: $v(t) = \dot{r} = (\dot{x}, \dot{y})$
- Speed: $\dot{s} = |v|$
- Acceleration: $a(t) = \dot{v} = \ddot{r} = (\ddot{x}, \ddot{y})$

Particle kinematics 2D



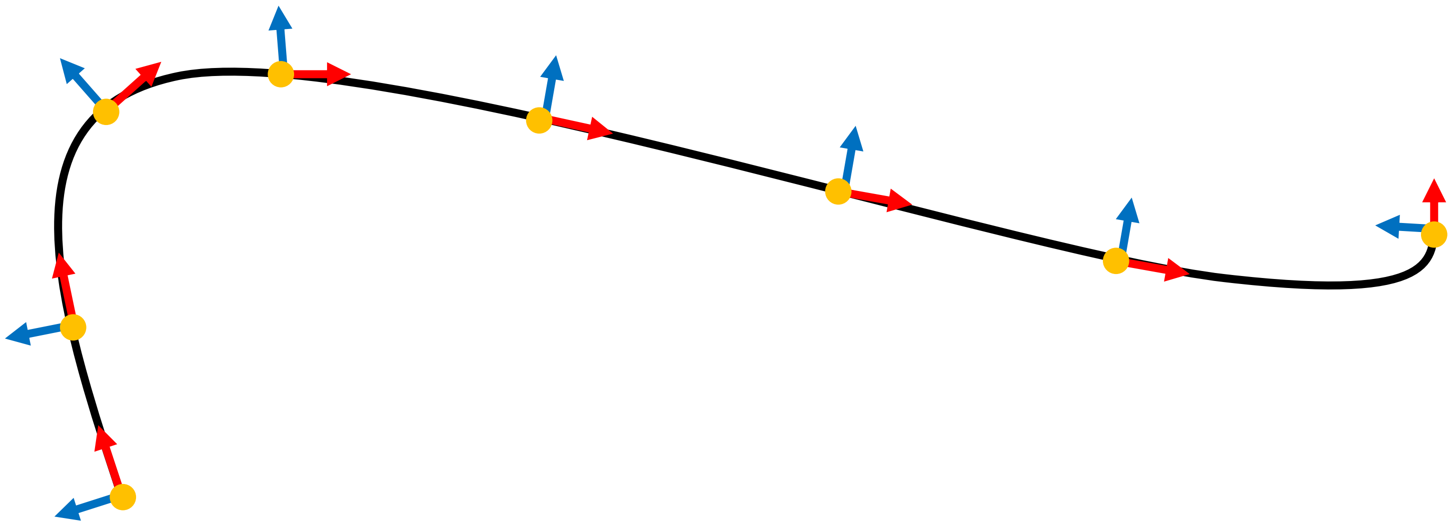
Particle kinematics 2D

- Tangent is $T(t) = \frac{v}{|v|}$



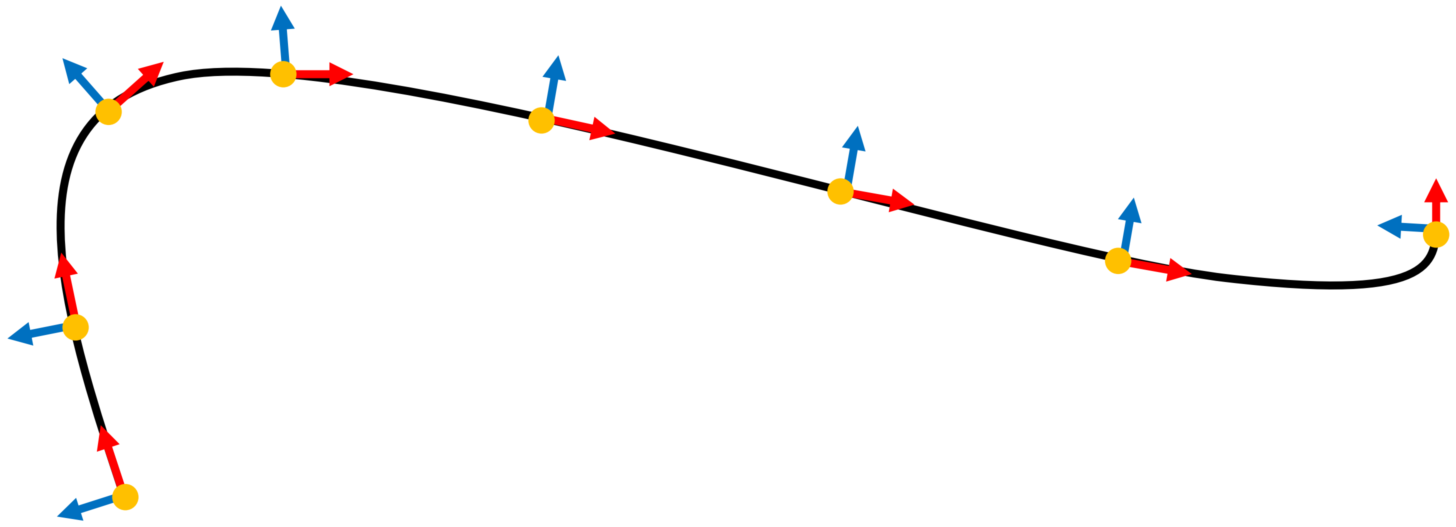
Particle kinematics 2D

- Tangent is $T(t) = \frac{v}{|v|}$
- Normal is $\frac{dT}{ds} = kN(t)$



Particle kinematics 2D

- Tangent is $T(t) = \frac{v}{|v|}$
- Normal is $\frac{dT}{ds} = kN(t)$
- r, T, N is the **moving frame** (also Frenet Frame) of the particle (or body in space)



Particle kinematics 3D

- Position at time t is

$$r(t) \; = \; x(t) \; + \; y(t) \; + \; z(t)$$

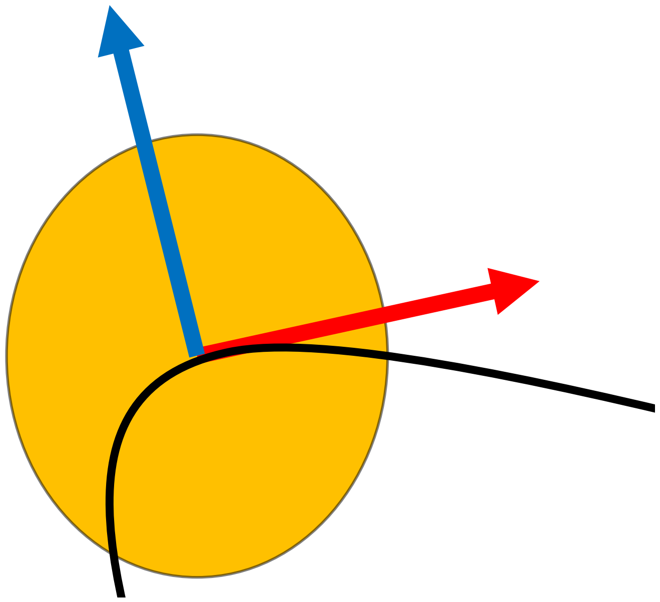
or $(x(t), y(t), z(t))$

Particle kinematics 3D

- Velocity at time t: $v(t) = \dot{r} = (\dot{x}, \dot{y}, \dot{z})$
- Speed: $\dot{s} = |v|$
- Acceleration: $a(t) = \dot{v} = \ddot{r} = (\ddot{x}, \ddot{y}, \ddot{z})$

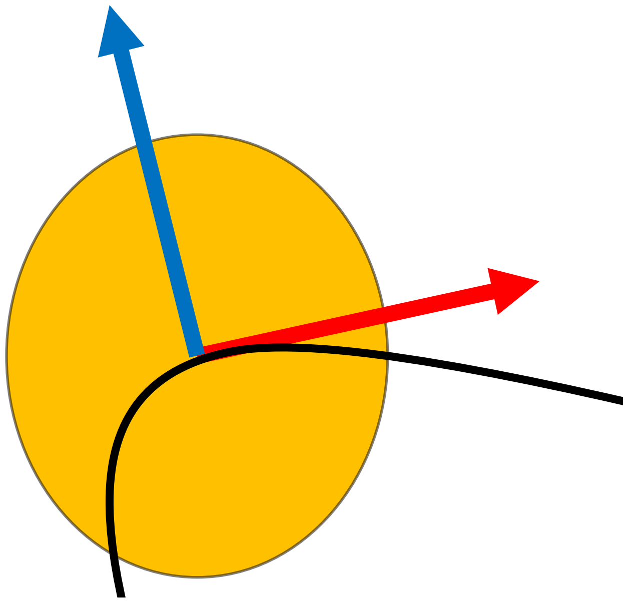
Particle kinematics 3D

- Tangent is $T(t) = \frac{v}{|v|}$



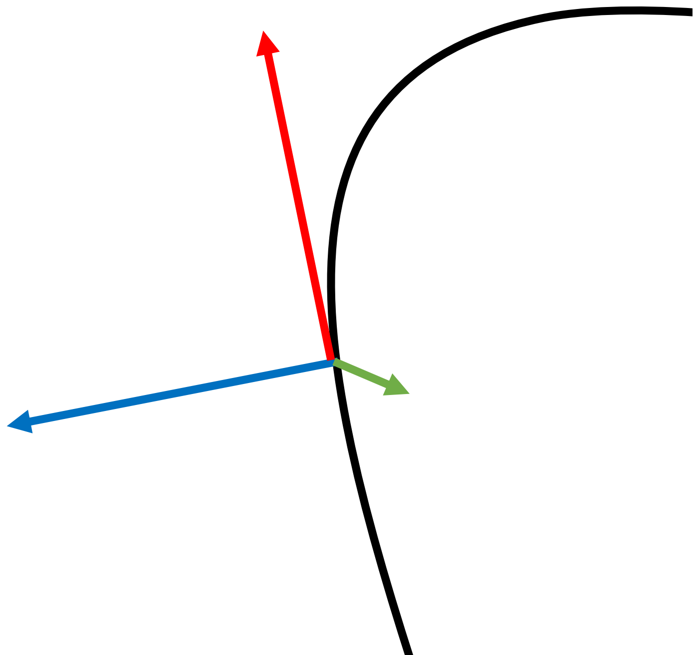
Particle kinematics 3D

- Tangent is $T(t) = \frac{v}{|v|}$
- We have an infinite set of possible vector normals to T



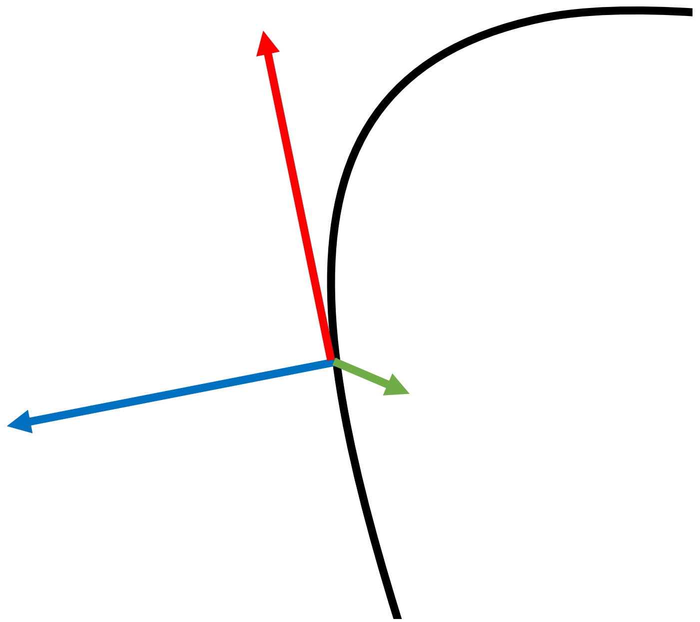
Particle kinematics 3D

- Binormal is $B = T \times N$



Particle kinematics 3D

- Binormal is $B = T \times N$
- r, T, N, B is the **moving frame** of the particle (or body in space)



Rigid Body Rotation

- $R = [T\ N\ B]$ put in matrix form is the **rotation matrix** of the body

Rigid Body Rotation

- $R = [T\ N\ B]$ put in matrix form is the **rotation matrix** of the body
- $r(t) = R(t)r_0 + x(t)$ where $x(t)$ is the position of the **center** of the body

Rigid Body Rotation

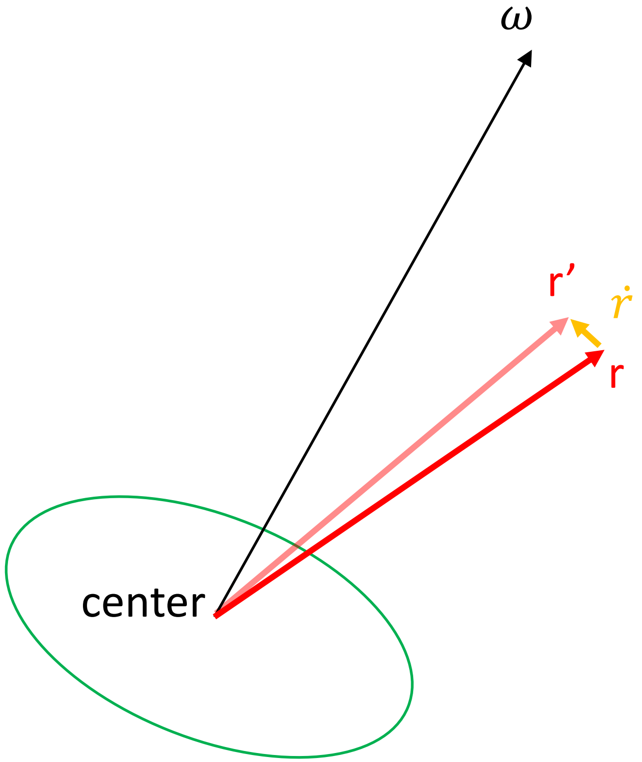
- $R = [T\ N\ B]$ put in matrix form is the **rotation matrix** of the body

- $r(t) = R(t)r_0 + x(t)$ where $x(t)$ is the position of the **center** of the body

- $\omega(t)$, a vector, is the **angular velocity** of the body
  - Its direction is the rotation axis
  - Its magnitude is in rad/s

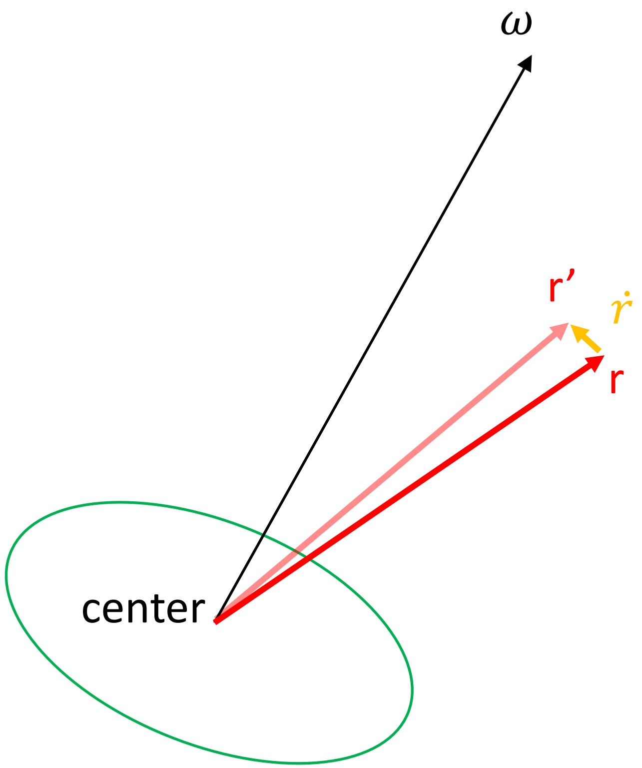
Rigid Body Rotation

- Calculate $\dot{r}(t)$



Rigid Body Rotation

- Calculate $\dot{r}(t)$



$$\dot{r} = \omega(t) \times r(t)$$

Rigid Body Rotation

- We now consider the columns of the rotation matrix
- We compute $\dot{T} = \omega(t) \times T, \dot{N} = \omega(t) \times N$ and $\dot{B} = \omega(t) \times B$

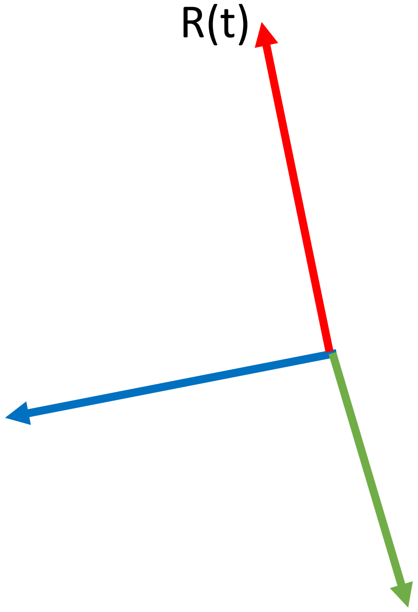
Rigid Body Rotation

- We now consider the columns of the rotation matrix

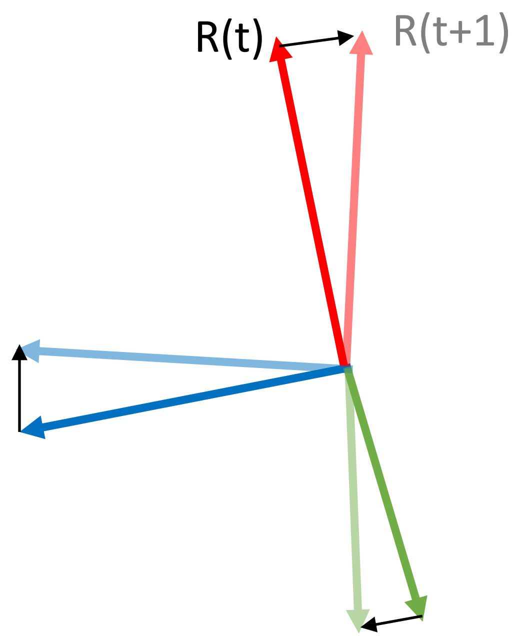
- We compute $\dot{T} = \omega(t) \times T$, $\dot{N} = \omega(t) \times N$ and $\dot{B} = \omega(t) \times B$

- $\dot{T}$, $\dot{N}$ and $\dot{B}$ are the velocities of the axes of the moving frame (the columns of $\dot{R}$)

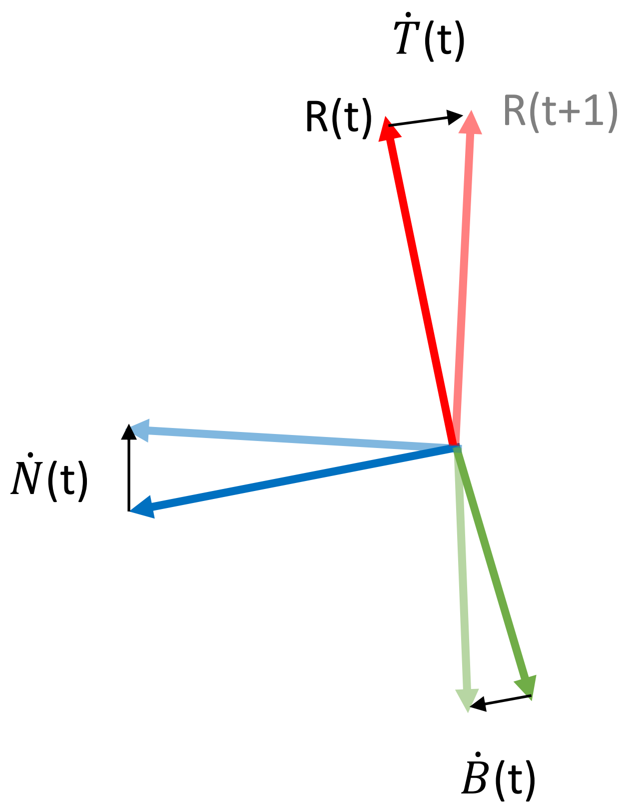
Geometric Interpretation of $\dot{R}$



Geometric Interpretation of $\dot{R}$



Geometric Interpretation of $\dot{R}$



Newton’s Laws

- **Inertia**, the tendency of an object to resist change of motion
- **Force**, the mechanism by which motion is changed

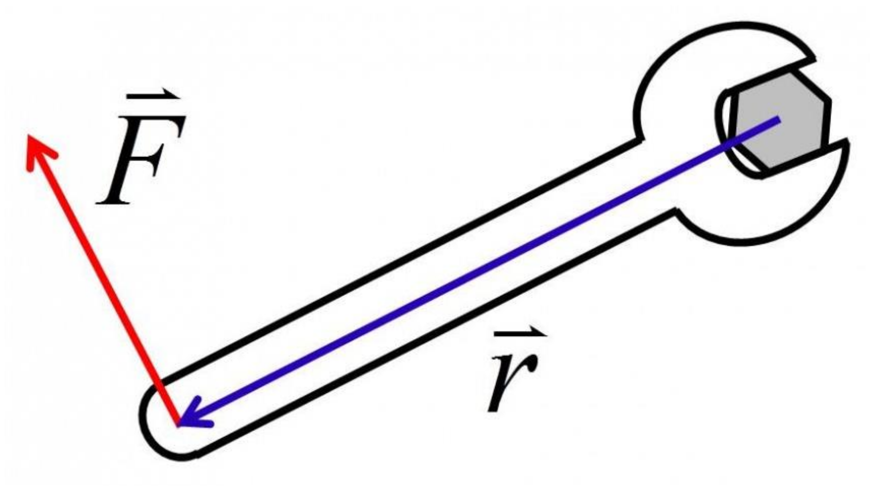
Newton’s Laws

- The **second law** states that in an inertial frame of reference, the vector sum of the forces F on an object is equal to the mass m of that object multiplied by the acceleration a of the object: F = ma. (It is assumed here that the mass m is constant – see below.) Mass is assumed to be always constant, so

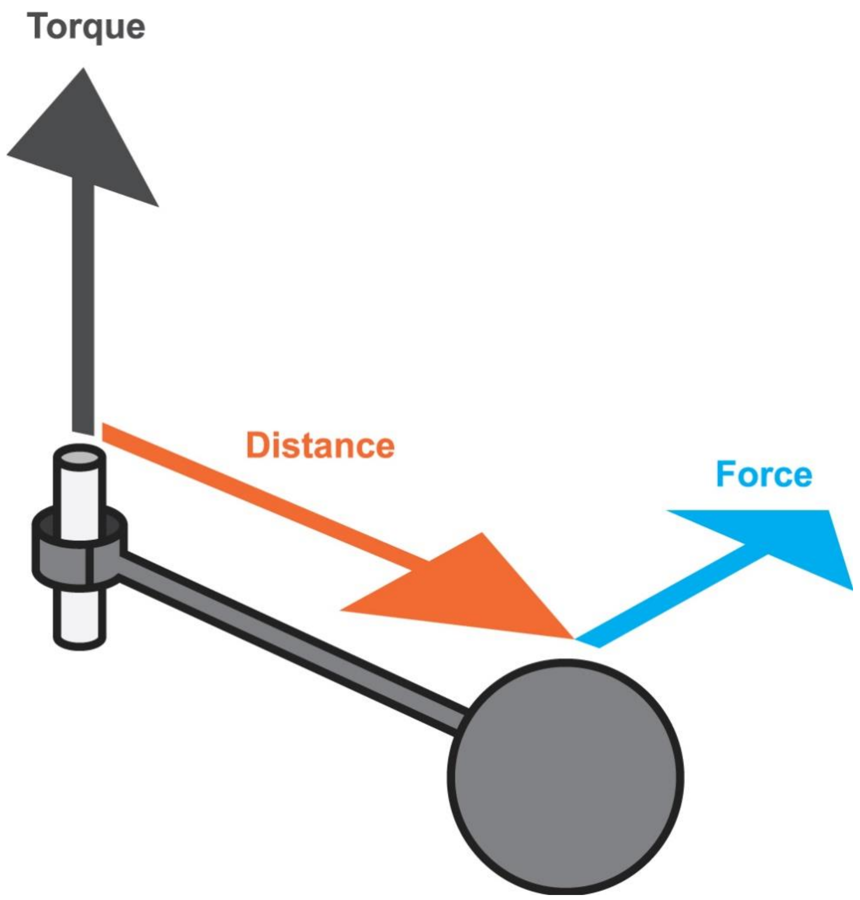
$$F = \frac{d}{dt}(mv) = ma$$

From Force to Torque

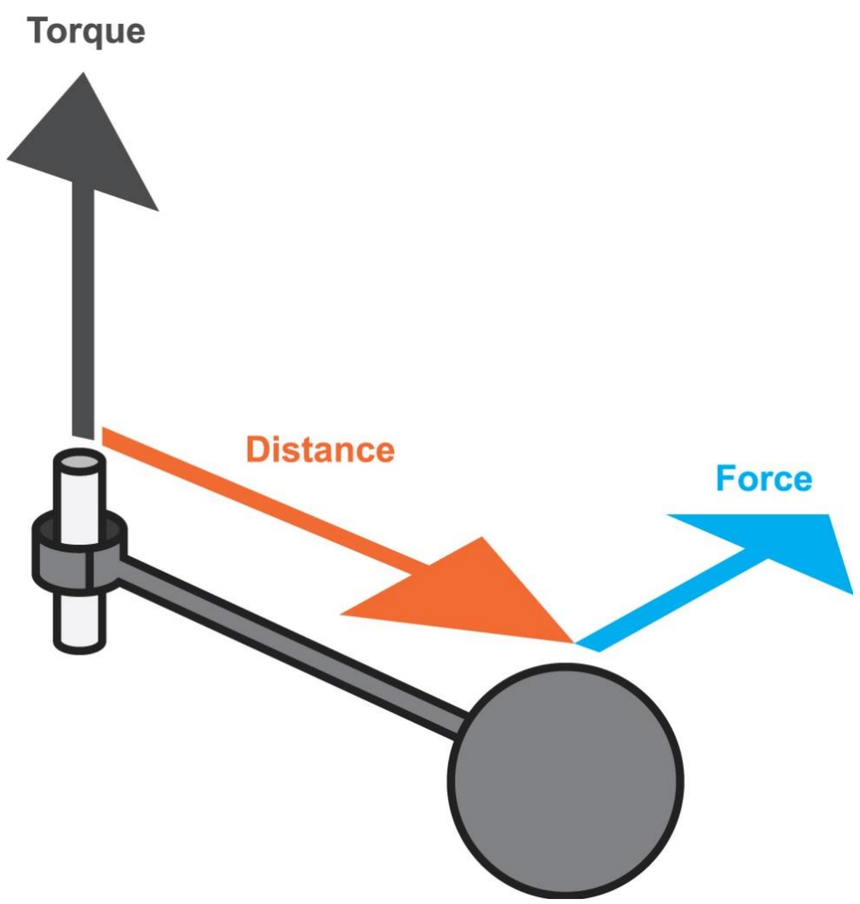
- Exert a force on the end of the wrench, the nut turns
- The longer the wrench, the easier (but slower) the nut turns



Torque τ



Torque τ



$$\tau = r \times F$$

Torque

- The ease of turning is proportional to the length of the wrench and the force applied
- This product is referred to as torque or moment of force
- Torque is defined as $\tau = r \times F$
  - Direction of torque is axis of rotation
  - Length of torque is in rad/s

Multiple Torques

- Multiple torques (just like forces) are simply added together
- $\tau = \sum_i r_i \times F_i$ (discrete body)

Linear momentum

- How much linear motion does a body have?
- $p = mv = \sum_i m_i v_i$

Linear momentum

- How much linear motion does a body have?
- $p = mv = \sum_i m_i v_i$
- Linear momentum is conserved in a system (all bodies $dp/dt = 0$)

Linear momentum

- How much linear motion does a body have?
- $p = mv = \sum_i m_i v_i$
- Linear momentum is conserved in a system (all bodies $dp/dt = 0$)
- Force integrates linear momentum directly

$$\frac{dp}{dt} = \frac{d(mv)}{dt} = m\,\frac{dv}{dt} = ma = F$$

Angular momentum

- How much rotational motion does a body have?
- $L =$

Angular momentum

- How much rotational motion does a body have?
- $L = r \times p =$

Angular momentum

- How much rotational motion does a body have?
- $L = r \times p = mr \times v$

Angular momentum

- How much rotational motion does a body have?
- $L = r \times p = mr \times v = \sum_i m_i \, r_i \times vi$
- Right-hand rule of cross-product:
  - Angular momentum refers to the tendency of the body to rotate around a given axis, L
  - The longer the axis, the harder it is to stop the rotation

Angular momentum

- The derivative of the angular momentum is torque (when the body does not change shape, $dr/dt = 0$)

$$\frac{dL}{dt} = \tau$$

Center of mass

- In mechanical systems, each object can behave as if its mass is concentrated at a single point. The location of this point is called the **center of mass** of the object.
- We compute the center of mass by a weighted average of the body particles relative positions x_i and their respective masses m_i (M is the mass of the whole body)

$$\bar{x} = \sum_i \frac{m_i x_i}{M}$$

Force projection

- When an **external force** F_{ext} is applied to a body from some position r_f
- We use the center of mass to split the force between linear force and torque

$$F \; = Fext(F_{ext} \cdot (r_f \; - \bar{x})), \qquad \tau = \; F_{ext} \; \times \; (r_f \; - \bar{x})$$

The implementation of the model so far

- Linear motion:

x

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v
```

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v = v + dt*a
    a
```

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v = v + dt*a
  a = f /m
```

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v = v + dt*a
  a = f /m
```

- Angular motion:

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v = v + dt*a
  a = f /m
```

- Angular motion:

```
R =
```

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v = v + dt*a
  a = f /m
```

- Angular motion:

```
R = R + dt*dR
```

dR

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v = v + dt*a
a = f /m
```

- Angular motion:

```
R = R + dt*dR
dR = [omega*T, omega*N, omega*B]
```

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v = v + dt*a
a = f /m
```

- Angular motion:

```
R = R + dt*dR
dR = [omega*T, omega*N, omega*B]
omega?
```

The implementation of the model so far

- Linear motion:

```
x = x + dt*v
v = v + dt*a
a = f /m
```

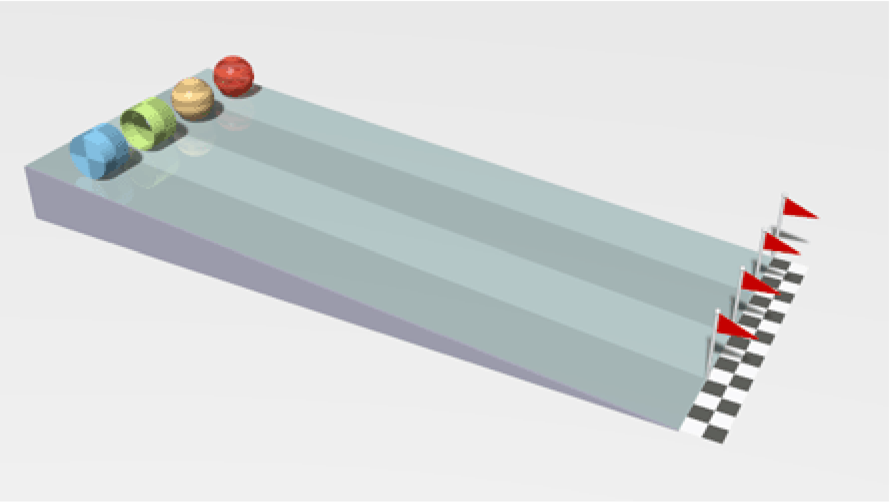
- Angular motion:

```
R = R + dt*dR
dR = [omega*T, omega*N, omega*B]
omega?
```

$$L = I\omega$$

Moments of Inertia

- How difficult is it to set an object into rotation around an axis?
- Rotational equivalent to mass for linear movement



Moment of Inertia

- Empirical studies for a **single** particle show that the moment of inertia is $\boldsymbol{mr^2}$, where m is the mass of the particle and r is its distance to the axis.

Moment of Inertia in 2D

- The more a particle weighs or the further from the center the harder to set the object into rotation
- In 2D the moment of inertia is a **single** number because we can only rotate in one plane

$$I = \sum_i m_i \,|(x_i\,, yi) - (\bar{x}, \bar{y})|^2 = \sum_i m_i\,(x_i{}^2 + yi^2)\; - mi(\bar{x}^2, \bar{y}^2)$$

Moment of Inertia in 2D

- The more a particle weighs or the further from the center the harder to set the object into rotation
- In 2D the moment of inertia is a **single** number because we can only rotate in one plane

$$I = \sum_i m_i \, |(x_i\,, yi) - (\bar{x}, \bar{y})|^2 = \sum_i m_i \, (x_i{}^2 + yi^2) \, - mi(\bar{x}^2, \bar{y}^2)$$

$$\omega = I^{-1} L$$

Mass matrix in 3D

- $L_i = ri \times miv_i$

Mass matrix in 3D

- $L_i = ri \times miv_i = m_i r_i \times (\omega \times r_i)$

Mass matrix in 3D

- $L_i = ri \times miv_i = m_i r_i \times (\omega \times r_i) = Ji\omega$

Mass matrix in 3D

- $L_i = ri \times mivi = m_i r_i \times (\omega \times ri) = Ji\omega$

- $J_i = mi \begin{bmatrix} {y_i}^2 + zi^2 & -xiyi & -xizi \\ -xiyi & {x_i}^2 + zi^2 & -y_i z_i \\ -xizi & -y_i z_i & {y_i}^2 + zi^2 \end{bmatrix}$

Mass matrix in 3D

- $L_i \;=\; ri \;\times\; mivi \;=\; m_i r_i \;\times\; (\omega \times\; ri) \;=\; Ji\omega$

- $J_i = mi \begin{bmatrix} {y_i}^2 + zi^2 & -xiyi & -xizi \\ -xiyi & {x_i}^2 + zi^2 & -y_i z_i \\ -xizi & -y_i z_i & {y_i}^2 + zi^2 \end{bmatrix}$
- $L_i = Ji\omega$, just like $p \;=\; mv$

Mass matrix in 3D

- $L_i \;=\; ri \;\times\; mivi \;=\; m_i r_i \;\times\; (\omega \times\; ri) \;=\; Ji\omega$
- $J_i = mi \begin{bmatrix} {y_i}^2 + zi^2 & -xiyi & -xizi \\ -xiyi & {x_i}^2 + zi^2 & -y_i z_i \\ -xizi & -y_i z_i & {y_i}^2 + zi^2 \end{bmatrix}$
- $L_i = Ji\omega$, just like $p \;=\; mv$
- For the whole body we sum all the J_i matrices of the particles
- $J = \sum_i J_i \,, L = J\omega$

Mass matrix in 3D

- $L_i = ri \times miv_i = m_i r_i \times (\omega \times r_i) = Ji\omega$
- $J_i = mi \begin{bmatrix} y_i^2 + zi^2 & -xiy_i & -xiz_i \\ -xiyi & x_i^2 + zi^2 & -y_i z_i \\ -xizi & -y_i z_i & y_i^2 + zi^2 \end{bmatrix}$
- $L_i = Ji\omega$, just like $p = mv$
- For the whole body we sum all the J_i matrices of the particles
- $J = \sum_i J_i , L = J\omega$
- J must be recalculated after rotation, because $r_i = Rr_0 + \bar{x}$

Whole Model - Position

- **Position**, integrated from velocity $\dot{x} = v$
- **Velocity**, derived from linear momentum $\dot{v} = p/m$
- **Linear momentum**, integrated from force

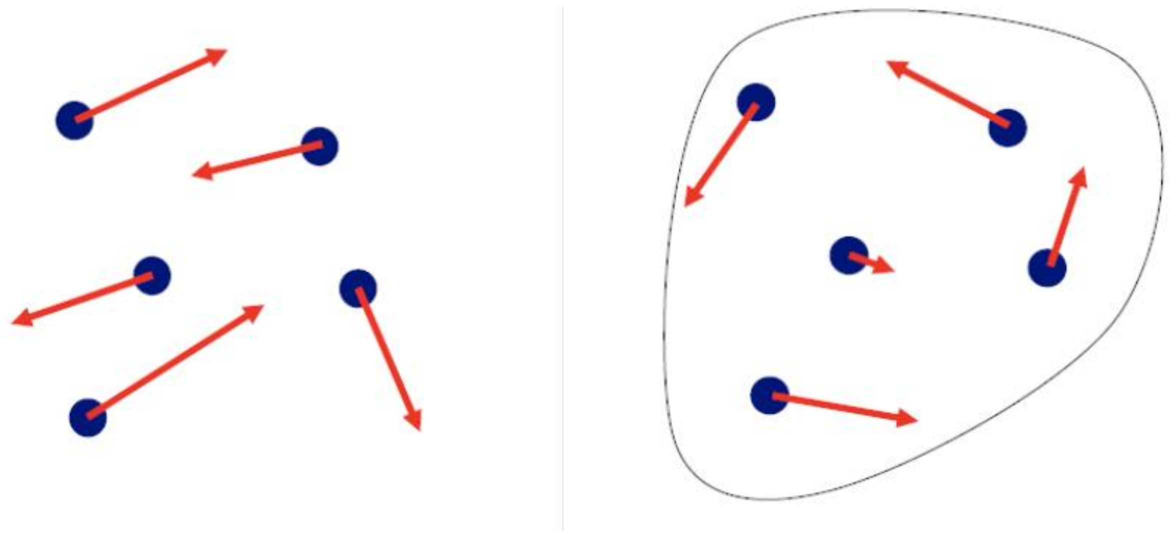
$$\dot{p} = Fex_t(F_{ext} \cdot (r_f - \bar{x}))$$

Whole Model - Rotation

- **Rotation**, integrated from angular velocity
$$\dot{R} = [\omega \times T \;\; \omega \times N \;\; \omega \times B]$$
- **Angular velocity**, derived from angular momentum $\omega = J^{-1}L$
- **Angular momentum**, integrated from torque

$$\dot{L} = \tau = F_{ext} \times (r_f - \bar{x})$$

From Particles to Rigid Bodies



- Particles
  - Only translation
  - **Linear velocity only**
  - 3 DoF

- Rigid bodies
  - 6 DoF (translation + rotation)
  - Linear velocity
  - **Angular velocity**