PFE 11

Palubicki

Computer simulations of biological patterns

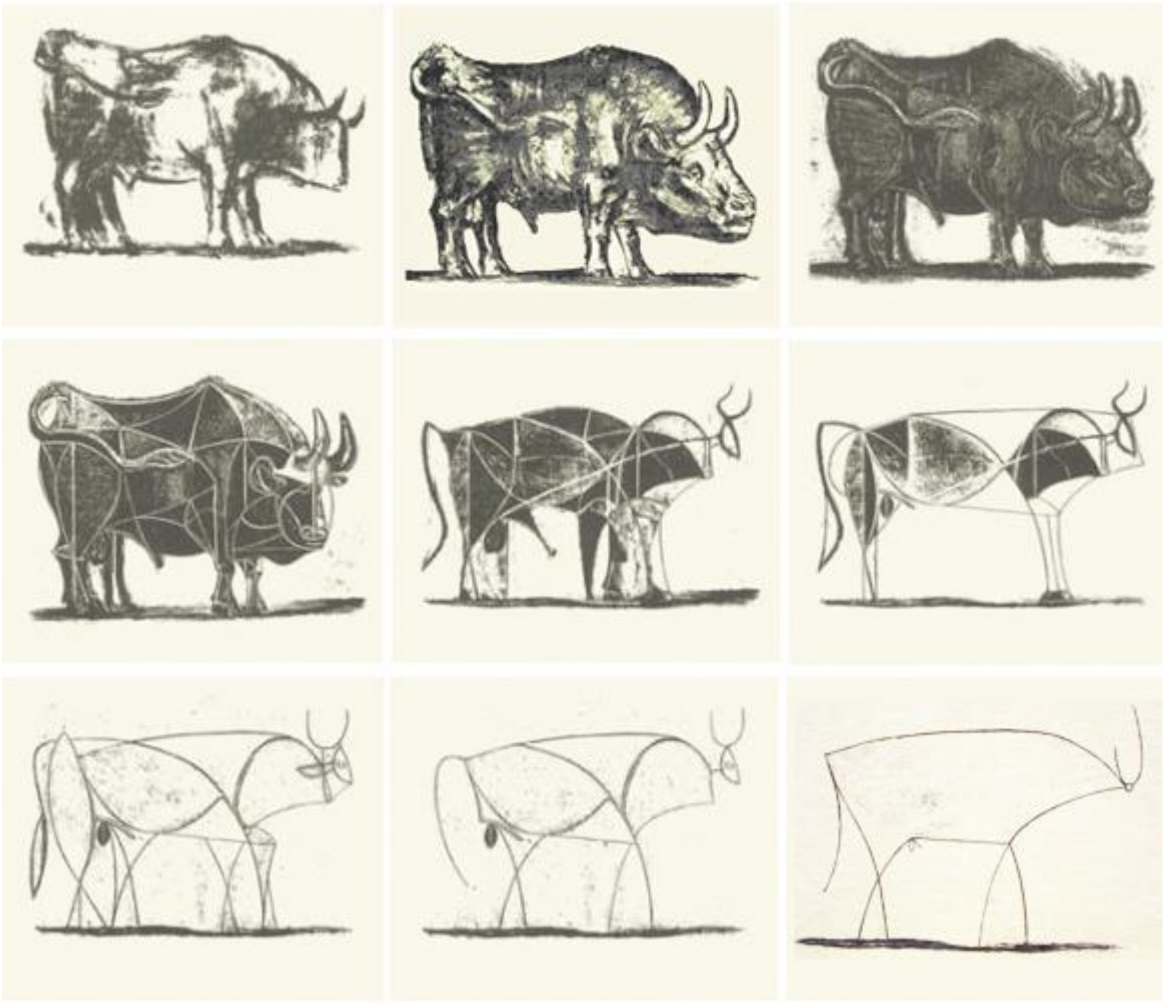
Synthetic tree

Real tree

Simulation of Biological Patterns

- **Simulation** is the operation of a mathematical **model** in time
- **Biological Model**: a simplified description of a biological pattern

Models of Biological Patterns



„Bull", Picasso

Models of Biological Patterns

- Computer models use formal descriptions of biological pattern formation with the help of mathematics and programming languages

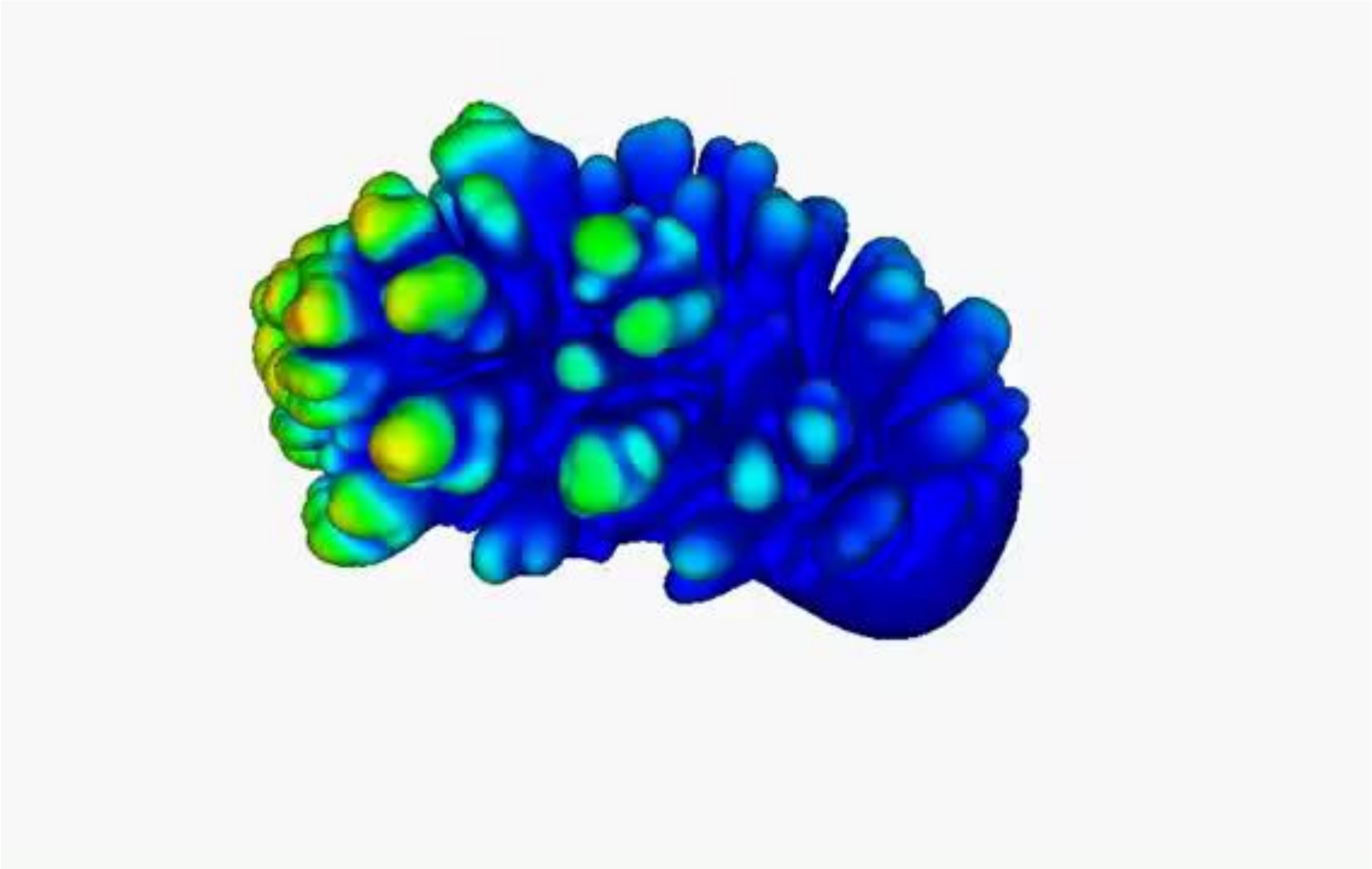
> *I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics; for men thus endowed seem to have an extra sense.*
>
> — Charles Darwin

Growth of a mango tree

Grechi et al., VPlants team, CIRAD, Montpellier

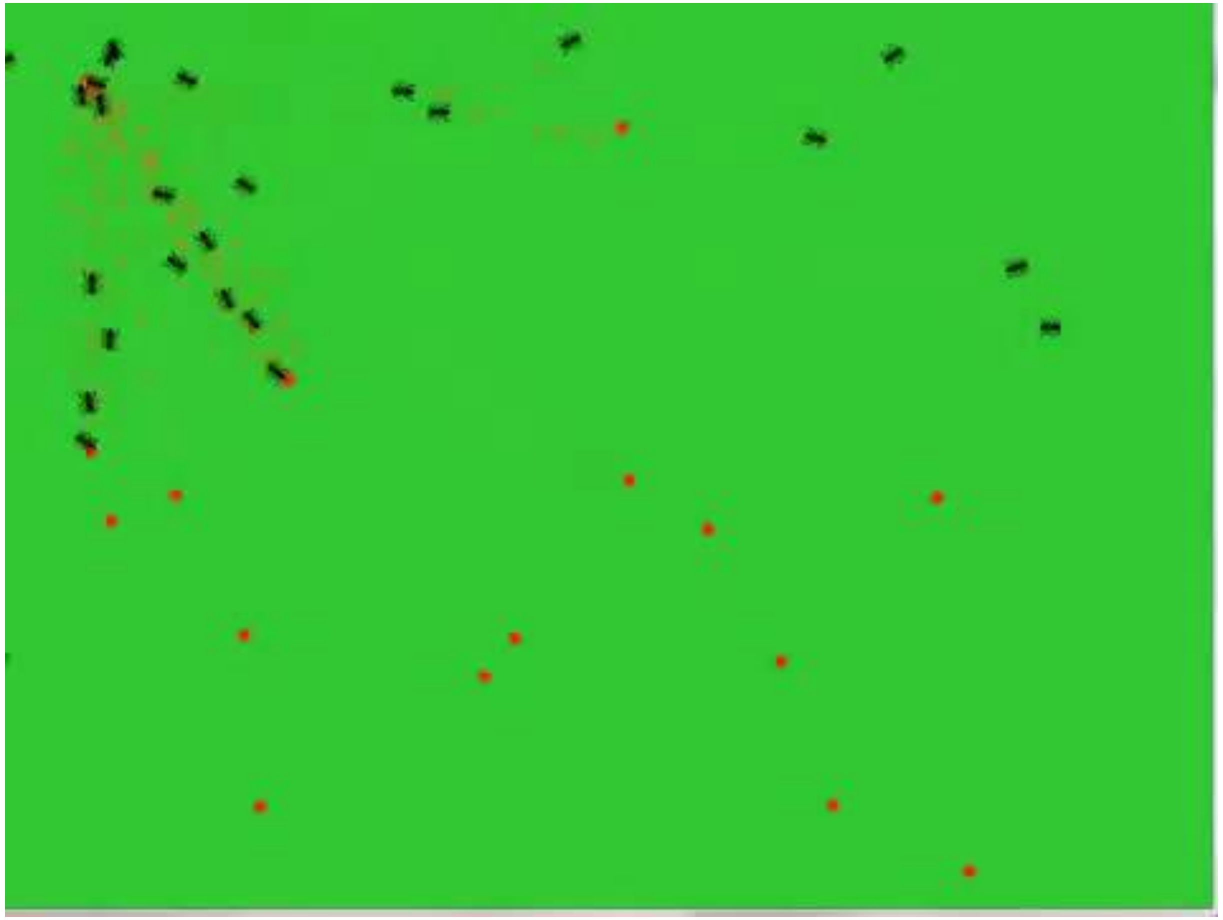
Fungi development

Coral growth

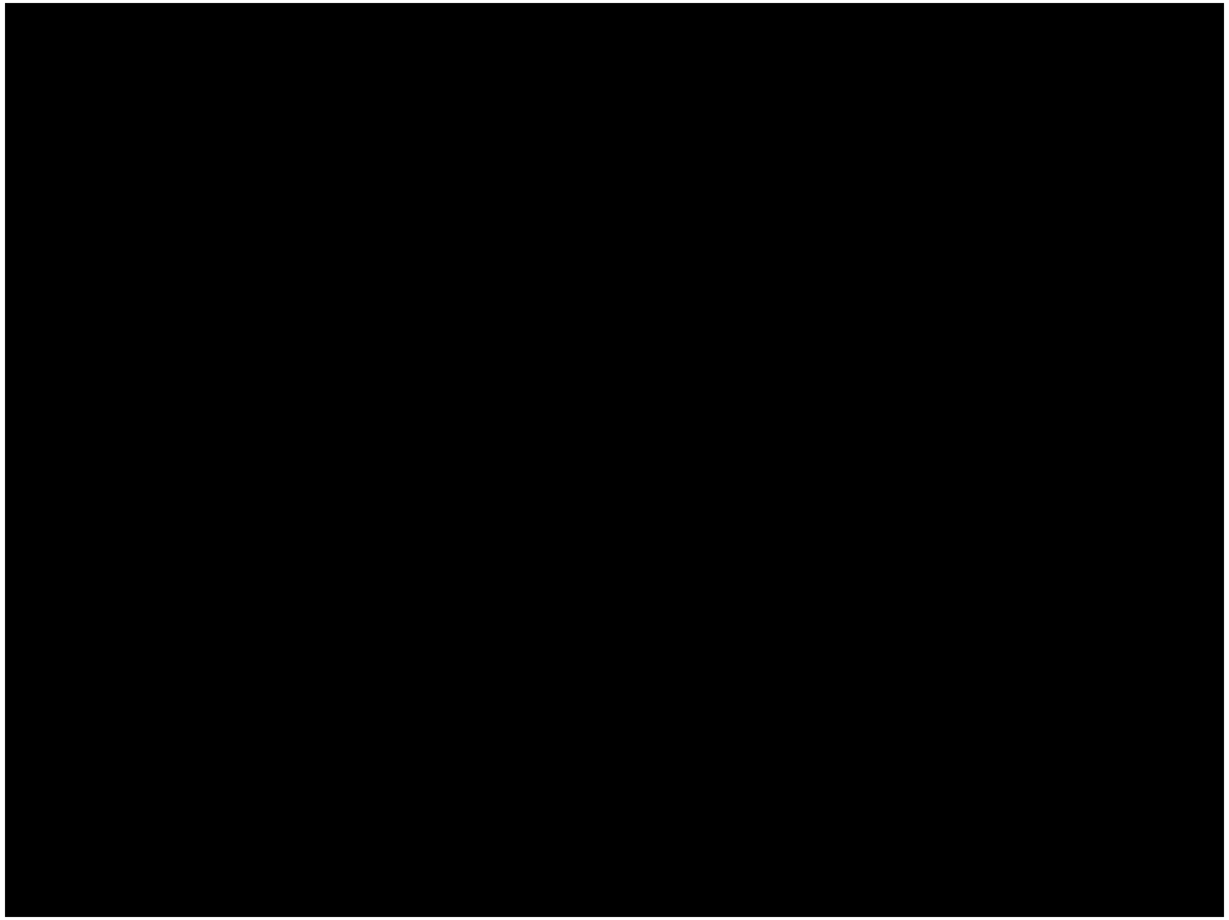


[Chindapol et al. 2013](#)

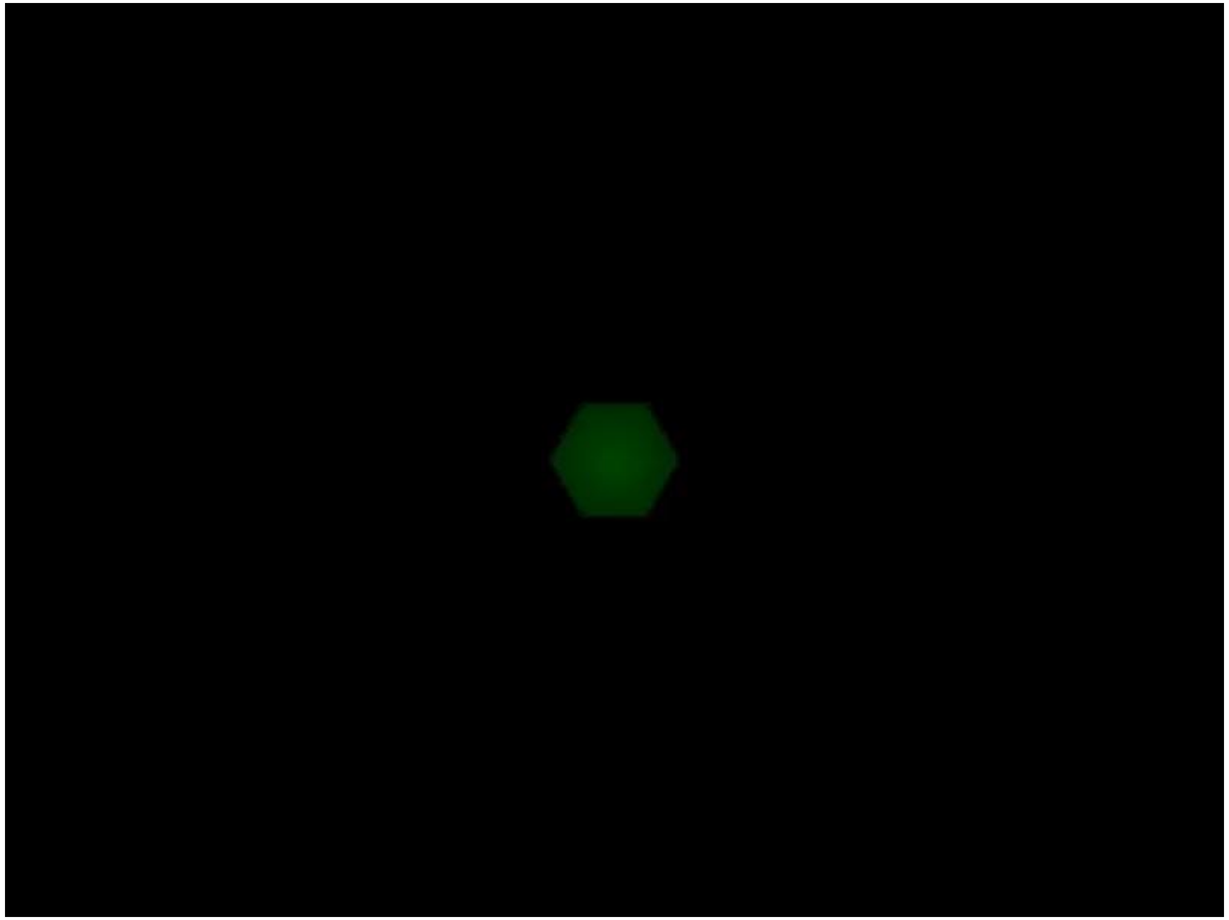
Ant trails



Bird flocking



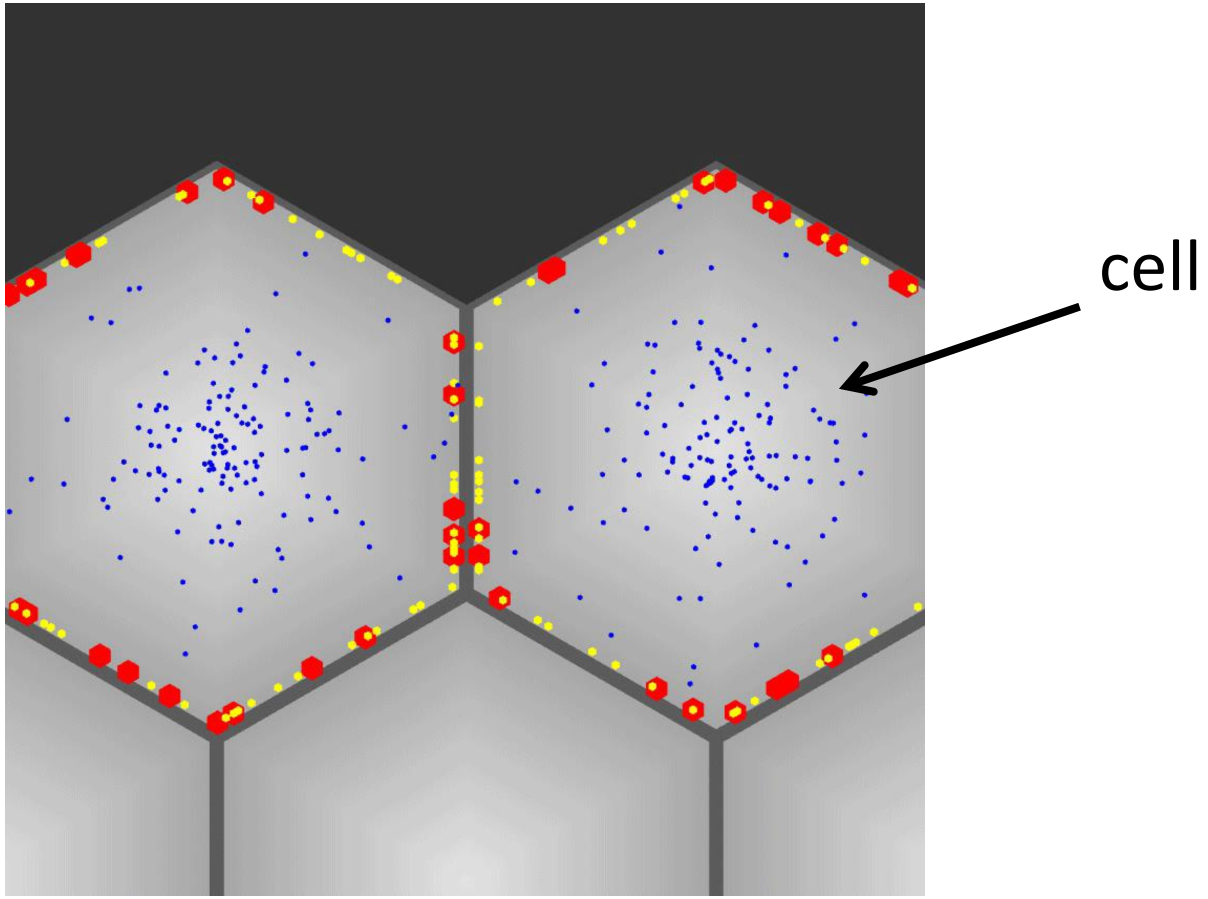
Plant Phyllotaxy



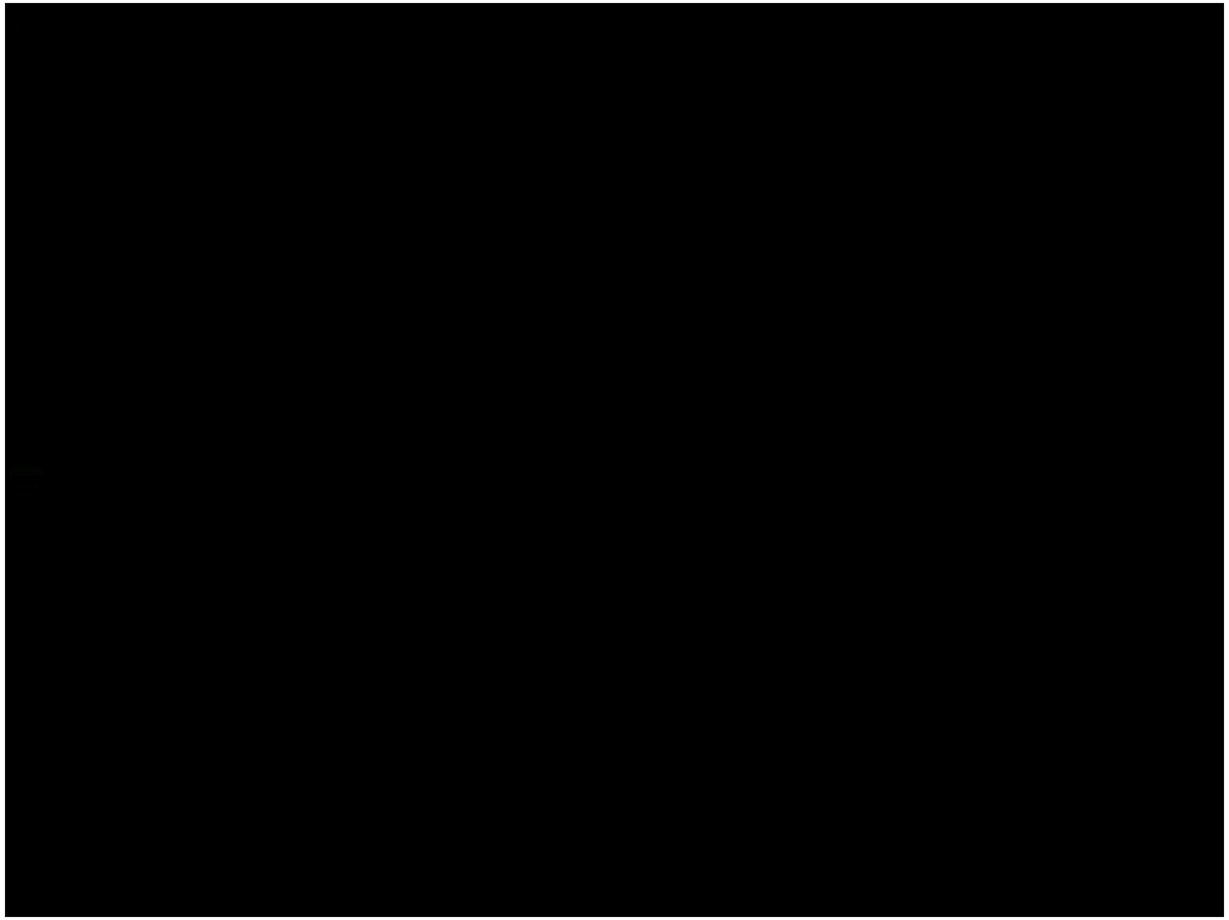
Smith et al. 2006

Protein transport in cells

- Auxin in cytoplasm
- Auxin in apoplast
- Efflux transporter
- Influx transporter

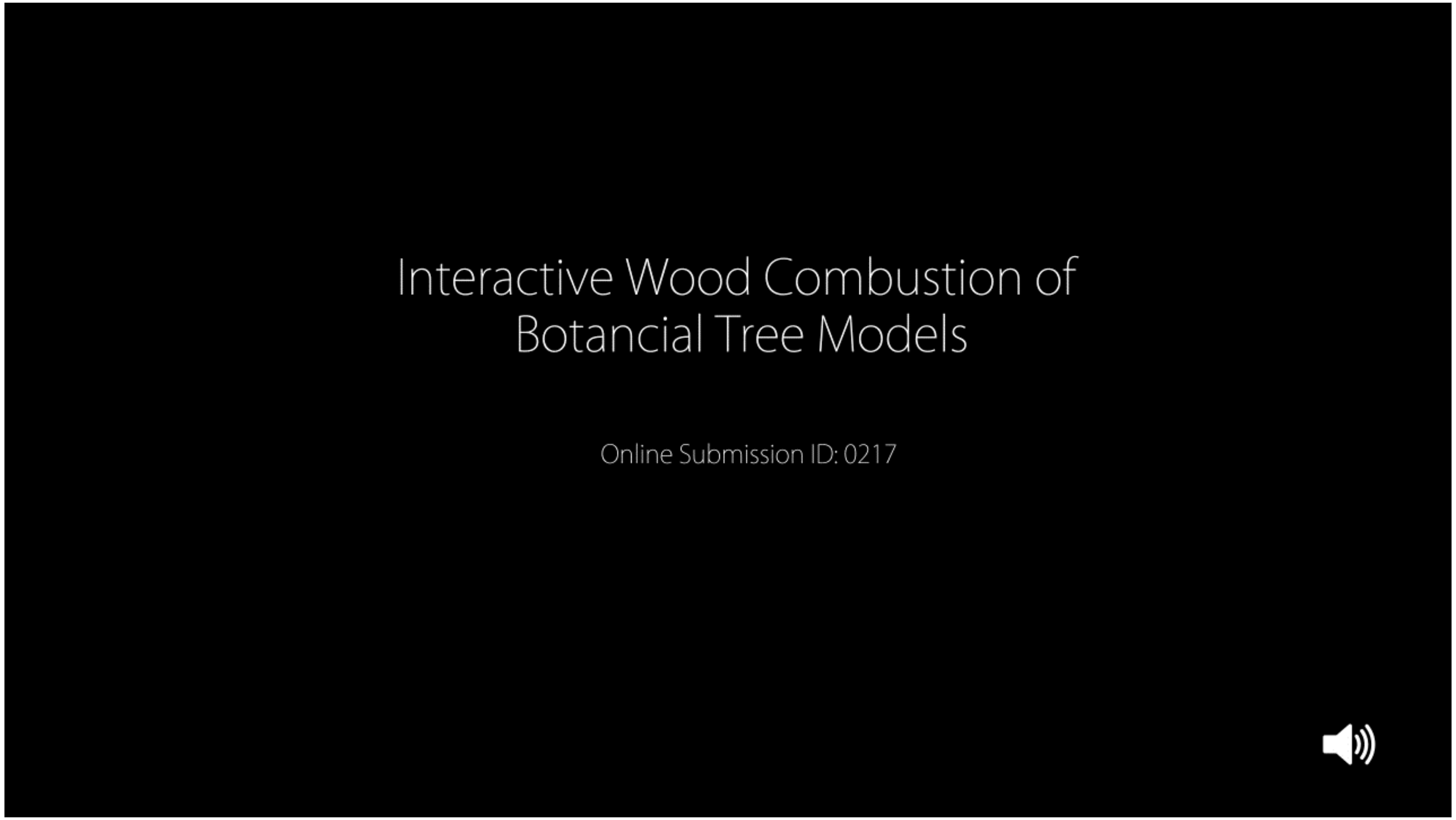


Protein folding



https://www.youtube.com/watch?v=iaHHgEoa2c8

Physical simulations



Modeling Scales

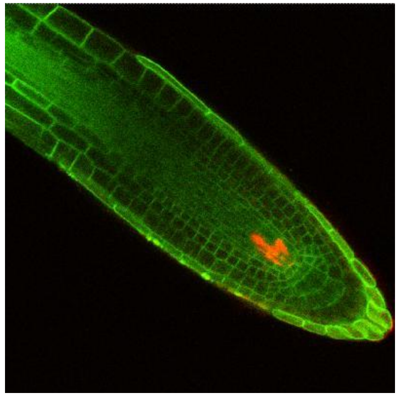
Population

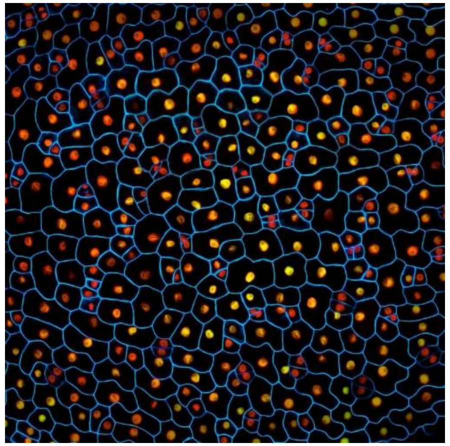
Organism

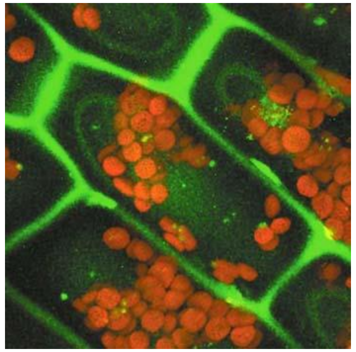
Organ

Tissue

Cell







Why Modeling?

- Visualization of concepts
- Communication of ideas
- Validation of hypotheses
- Generation of new hypotheses
- Predicting the future
- Artistic tools
- …

Different Modeling Methods

Vertex models

Formal grammars

Particle models

Cellular automata

Boolean nets

Agent-based models

Differential equation methods

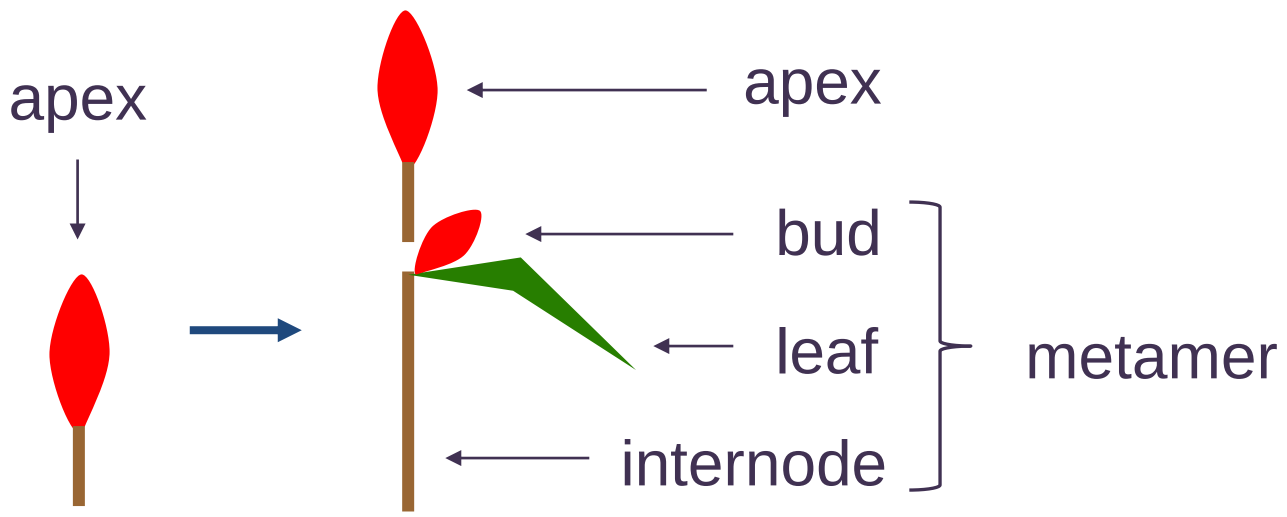
Finite element methods

Genetic algorithms

Formal Grammars

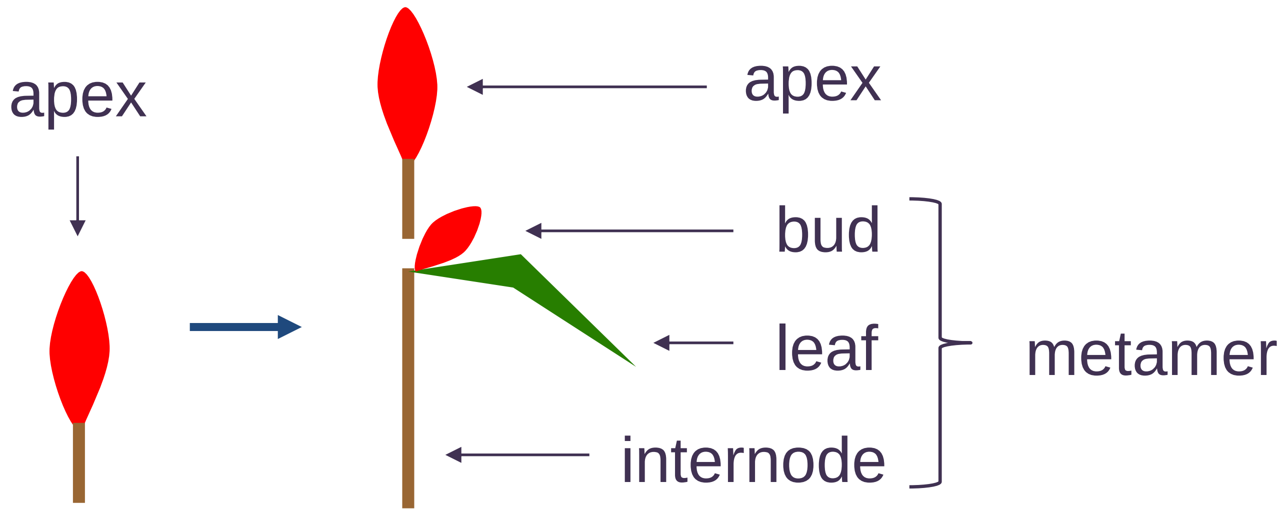
How to describe development mathematically?

The fundamental developmental scheme



rule

The fundamental developmental scheme



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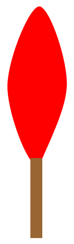

development

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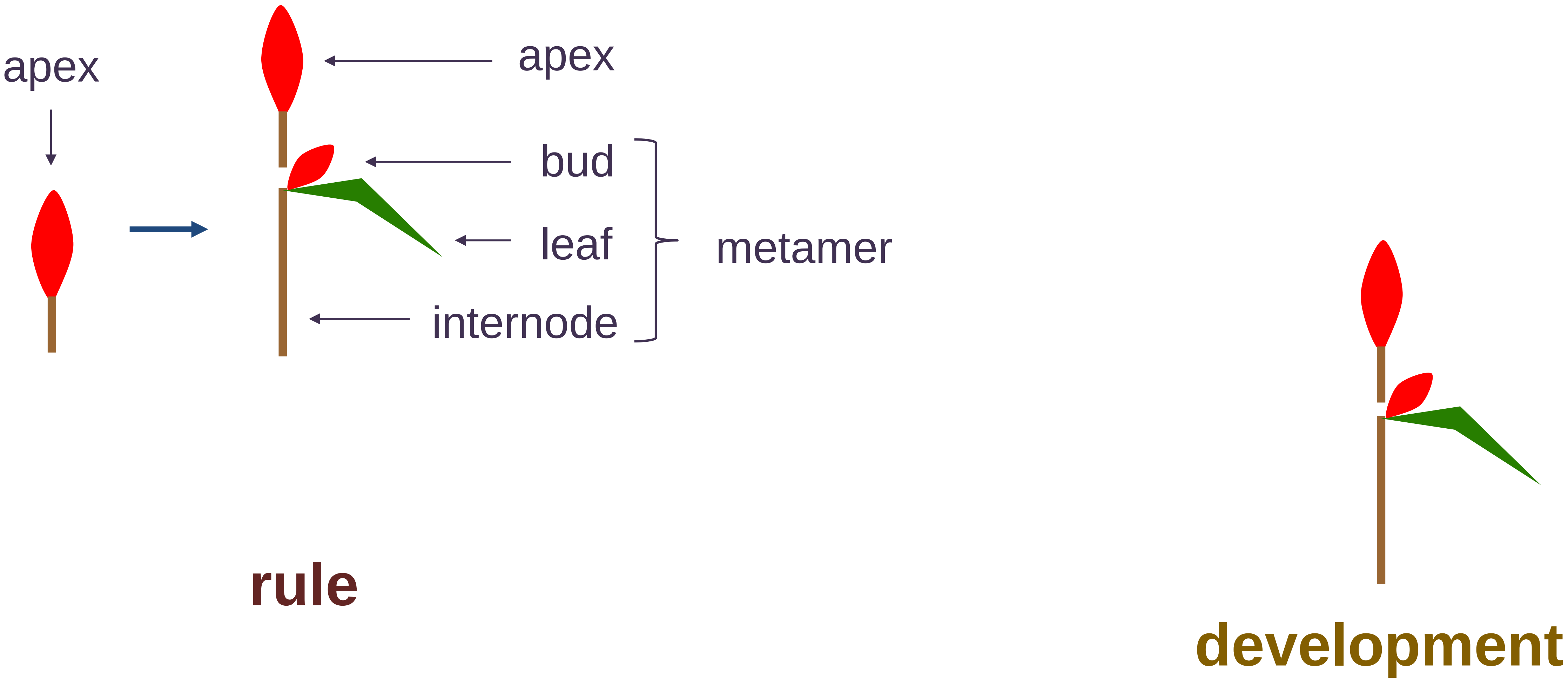

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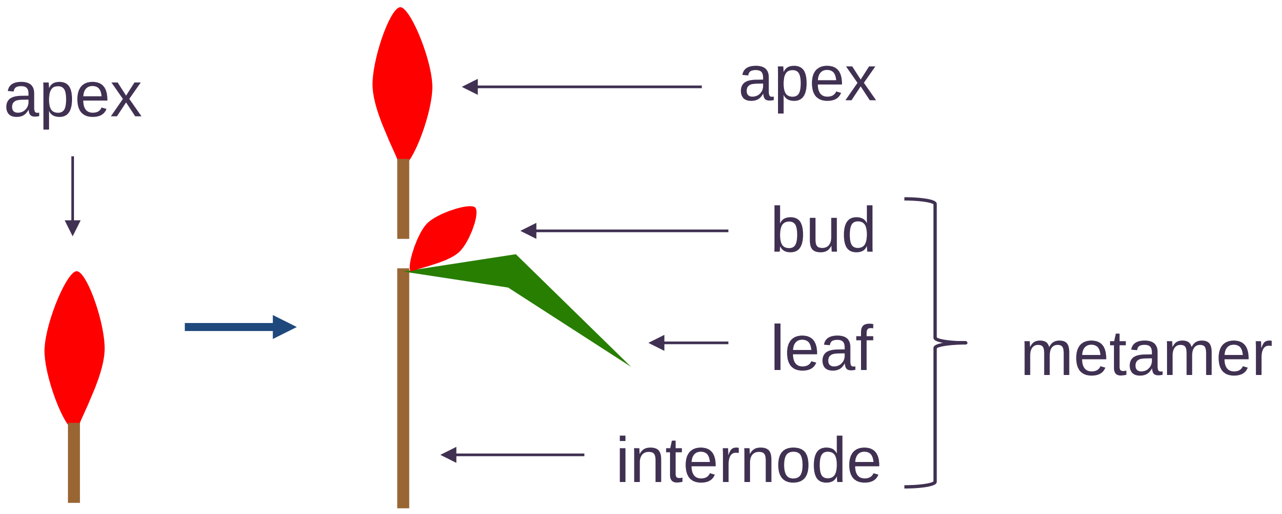

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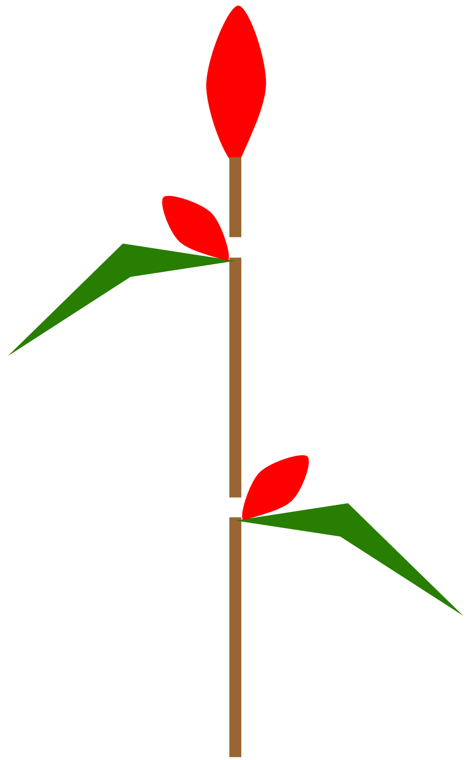

development

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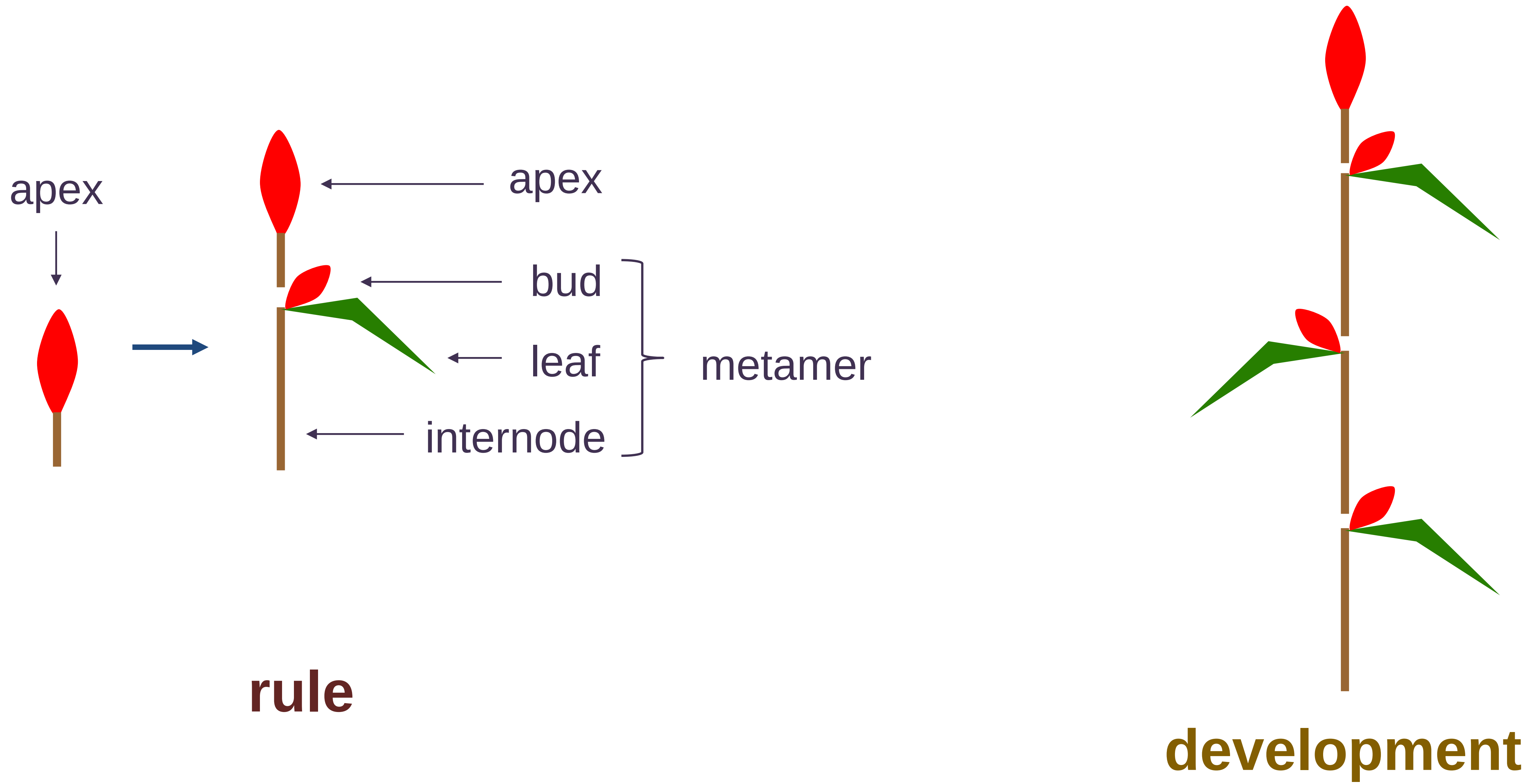
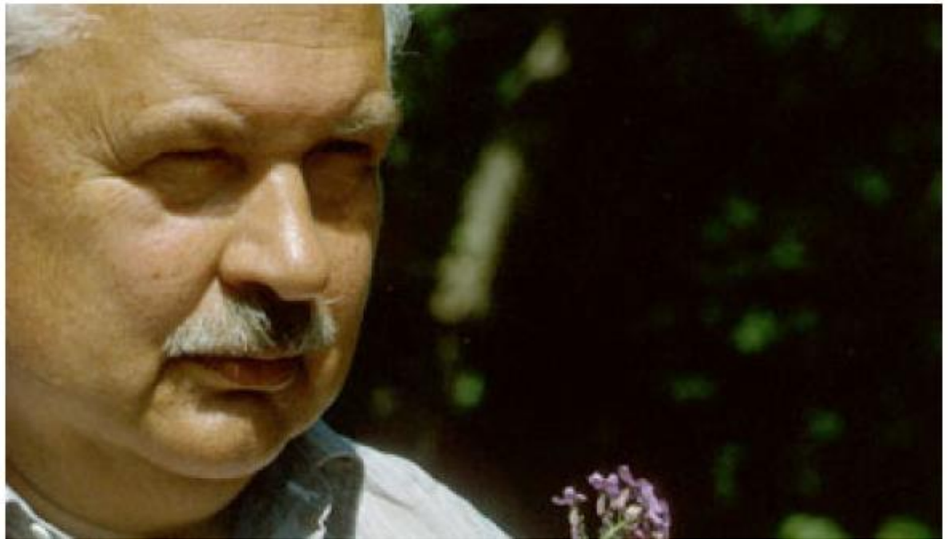
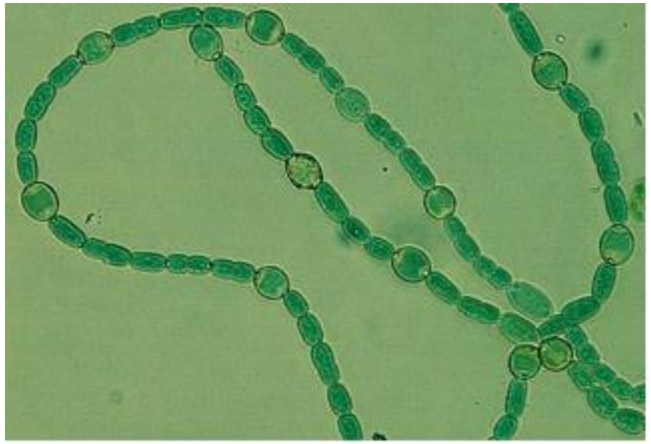


L-Systems: a language to describe growth

- Aristid Lindenmayer (1925-1989)
  - Anabaena Catenula
  - 1968 Lindenmayer systems – parallel string rewriting systems





L-Systems: a language to describe growth

- Later used to model plants

Przemysław Prusinkiewicz

[Algorithmic beauty of plants. P. Prusinkiewicz 1992]

Simplest case: D0L-Systems

- Formal grammar $G = (V,\ \omega,\ P)$
- V → alphabet of symbols containing elements that can be replaced
- ω → axiom (the initial state of the system)
- P → a set of production rules defining how symbols can be replaced

- Example:
  – $V = (A, M)$
  – $\omega = A$
  – $P = [(A \rightarrow MA), (M \rightarrow M)]$

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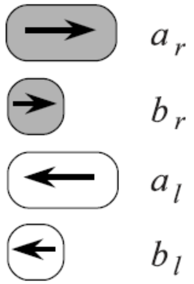
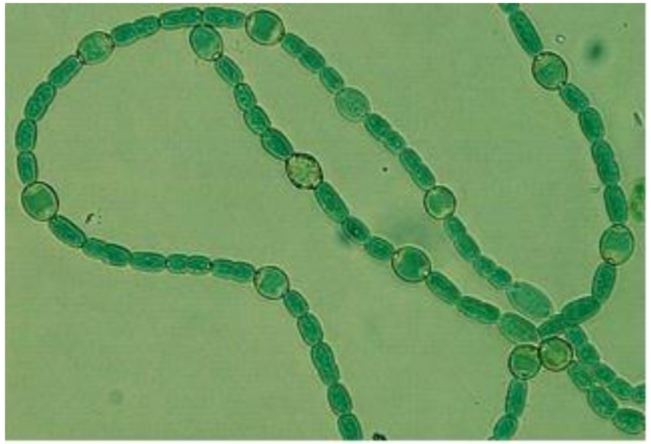
n=0: A
n=1: MA
n=2: MMA
n=3: MMMA

Anabaena Catenula: Model 1

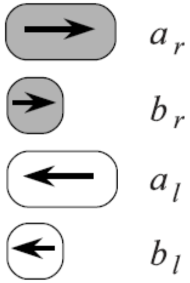


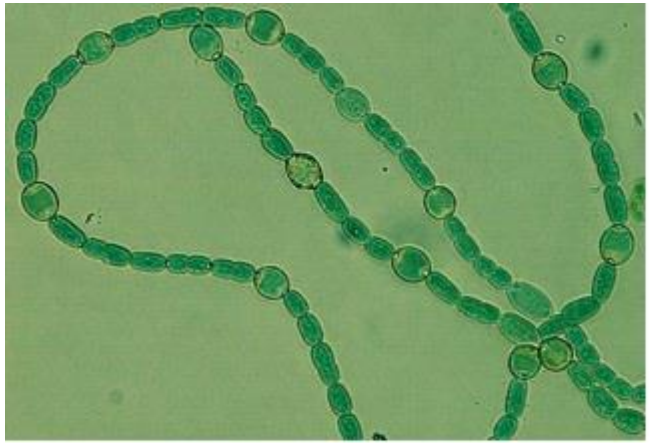


Anabaena Catenula: Model 1

$$p_1 : a_r \rightarrow a_l b_r$$

$$p_2 : a_l \rightarrow b_l a_r$$





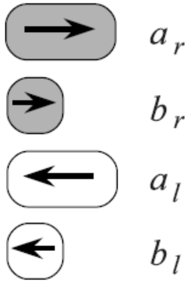
Anabaena Catenula: Model 1

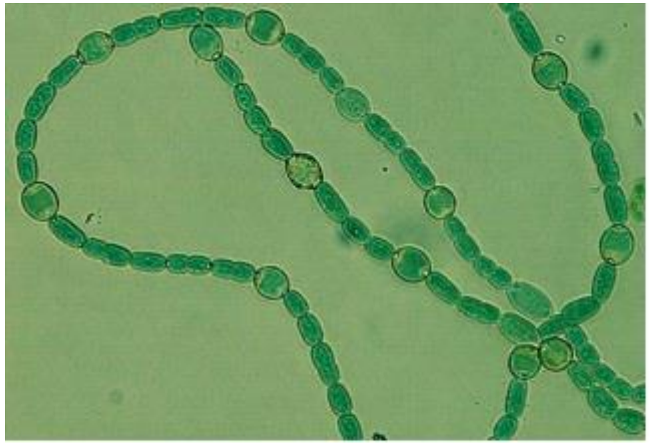
$p_1 : a_r \rightarrow a_l b_r$

$p_2 : a_l \rightarrow b_l a_r$

$p_3 : b_r \rightarrow a_r$

$p_4 : b_l \rightarrow a_l$





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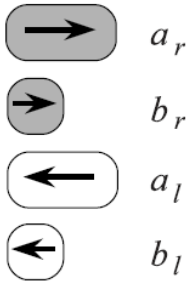

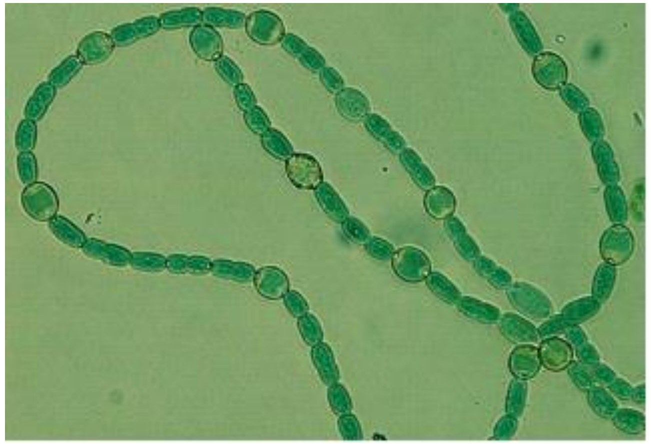

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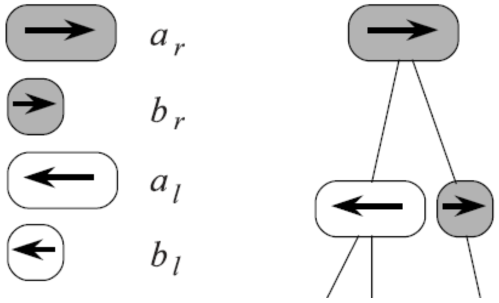

a_r

$a_l b_r$

Anabaena Catenula: Model 1

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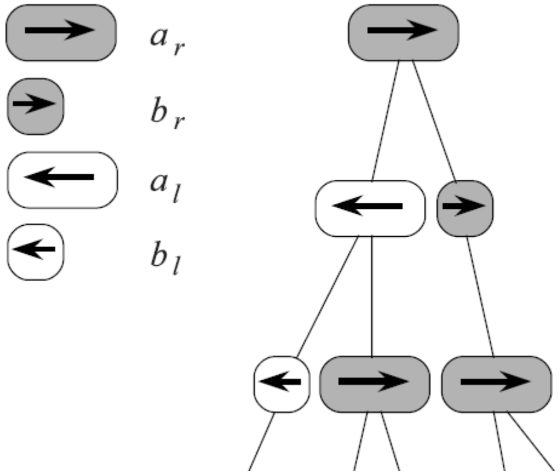

a_r

$a_l b_r$

$b_l a_r a_r$

Anabaena Catenula: Model 1

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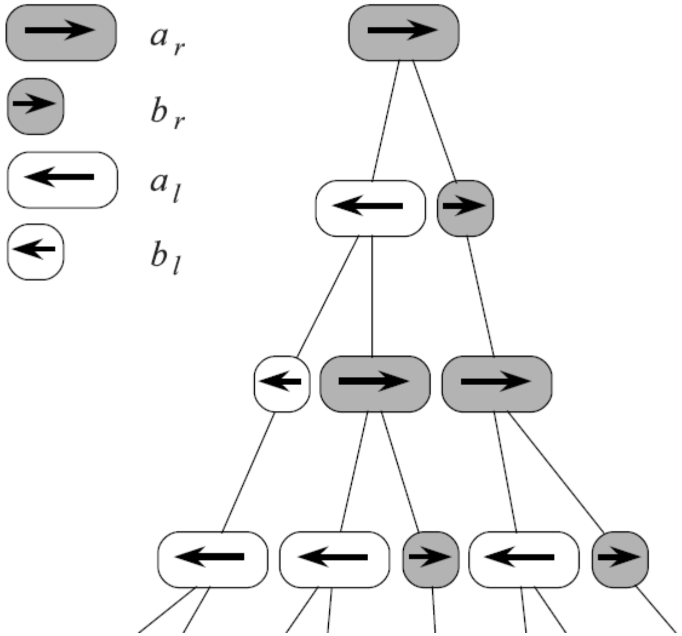

a_r

$a_l b_r$

$b_l a_r a_r$

$a_l a_l b_r a_l b_r$

Anabaena Catenula: Model 1

$\omega : a_r$

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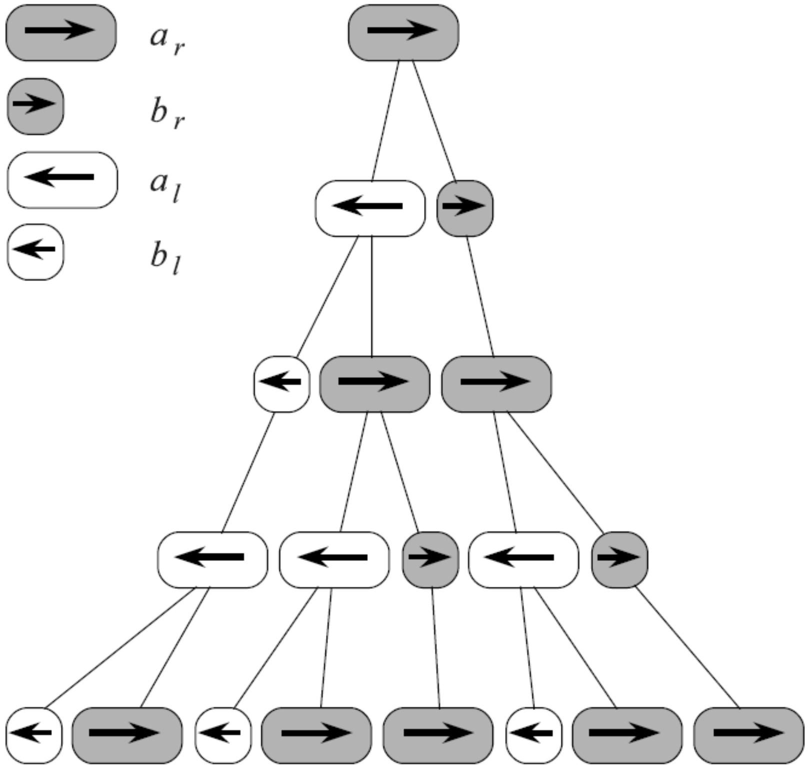


a_r

$a_l b_r$

$b_l a_r a_r$

$a_l a_l b_r a_l b_r$

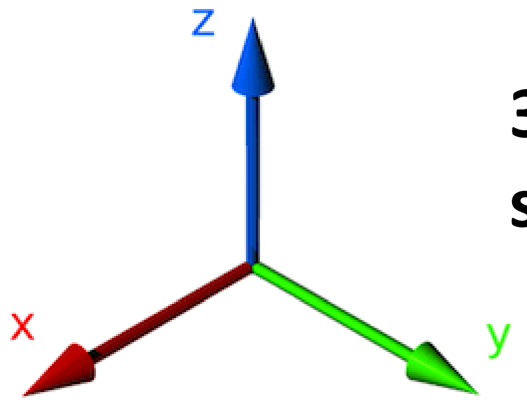
$b_l a_r b_l a_r a_r b_l a_r a_r$

Geometric interpretation of L-Systems

1D string

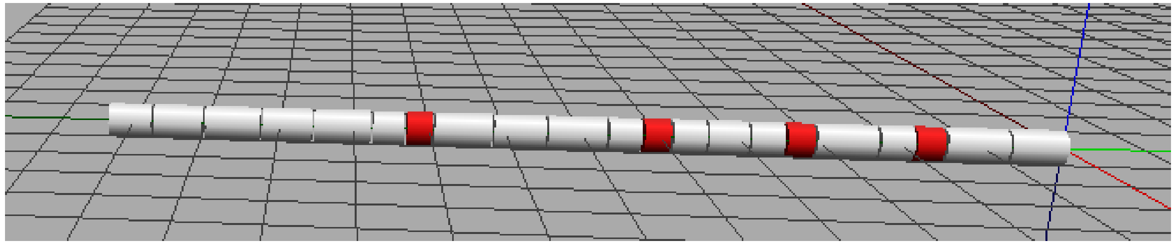
$b_l a_r b_l a_r a_r b_l a_r a_r$

+

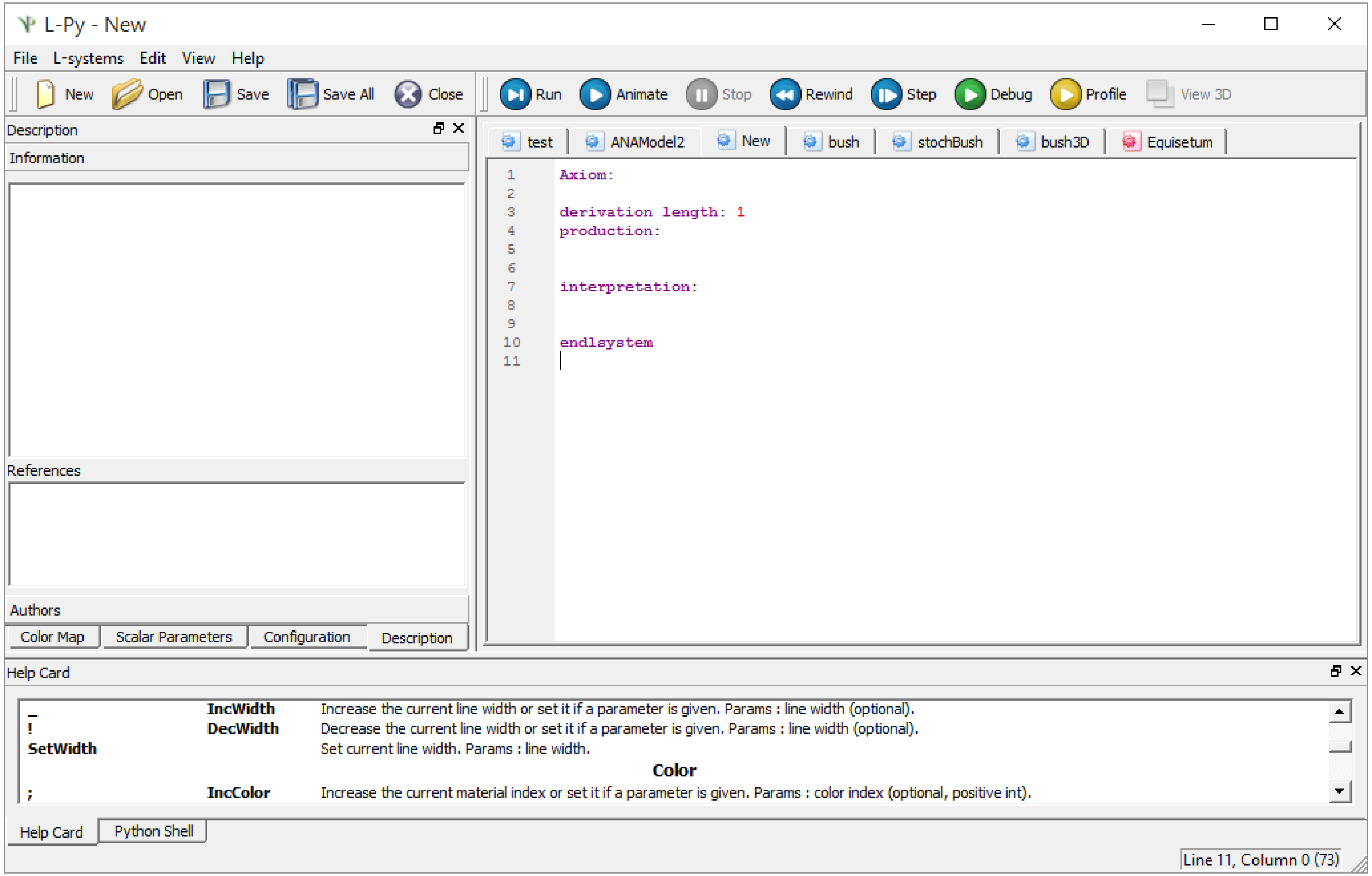


3D continuous space

geometric interpretation



L-Py



L-Py - New
File L-systems Edit View Help
New Open Save Save All Close
Run Animate Stop Rewind Step Debug Profile View 3D
Description
Information
References
Authors
Color Map Scalar Parameters Configuration Description
test ANAModel2 New bush stochBush bush3D Equisetum
Axiom:
derivation length: 1
production:
interpretation:
endlsystem
Help Card
_ IncWidth Increase the current line width or set it if a parameter is given. Params : line width (optional).
! DecWidth Decrease the current line width or set it if a parameter is given. Params : line width (optional).
SetWidth Set current line width. Params : line width.
Color
; IncColor Increase the current material index or set it if a parameter is given. Params : color index (optional, positive int).
Help Card Python Shell
Line 11, Column 0 (73)

L-Py

```
Axiom:

derivation length: 1
production:


interpretation:


endlsystem
```

L-Py: an L-system simulation framework for modeling plant architecture development based on a dynamic language.

L-Py (production)

- Production rule: A → AB

```
A: produce A B
```

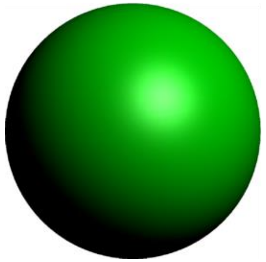
- Move in 3D space in positive x-direction:

```
f(length)
```

L-Py (interpretation)

- Drawing spheres:

```
@O(radius)
```



- Setting colors:

```
SetColor(range 0-255)
```

L-Py Example

```
# Leafy axis

from openalea.plantgl.all import *
from math import *

nb_vertical_segments = 8
length = 1      # in some units (= cm for example)
dl = 0.1
scaling = 2     # to dilate/contract the leaf

phi = 180       # Phyllotactic angle
h = 1           # height of an internode

module Leaf

Axiom: /(90) A(0)
derivation length: 10
production:

# try ^(-90)f(0.05)^(90) to translate the leaf on the periphery of the stem
A(n):
  produce F(h)/(phi)[^(-45)Leaf(scaling)]A(n+1)

interpretation:
maximum depth: 2

# Organ definitions
Leaf(x) --> ;(2)Sweep(nerve,section,length,dl,x,width)

endlsystem
```

Parametric L-Systems

$$A(x) \xrightarrow{x>0} A(x-1)$$

$$A(x) \xrightarrow{x=0} A(x+3)B(x+3)$$

$$A(3) \rightarrow A(2) \rightarrow A(1) \rightarrow \mathrm{A}(0) \rightarrow A(3)B(3)$$

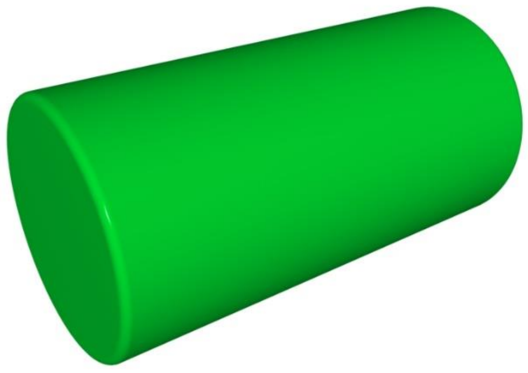
L-Py

```
A(x): if x > 0: produce A(x-1)
```

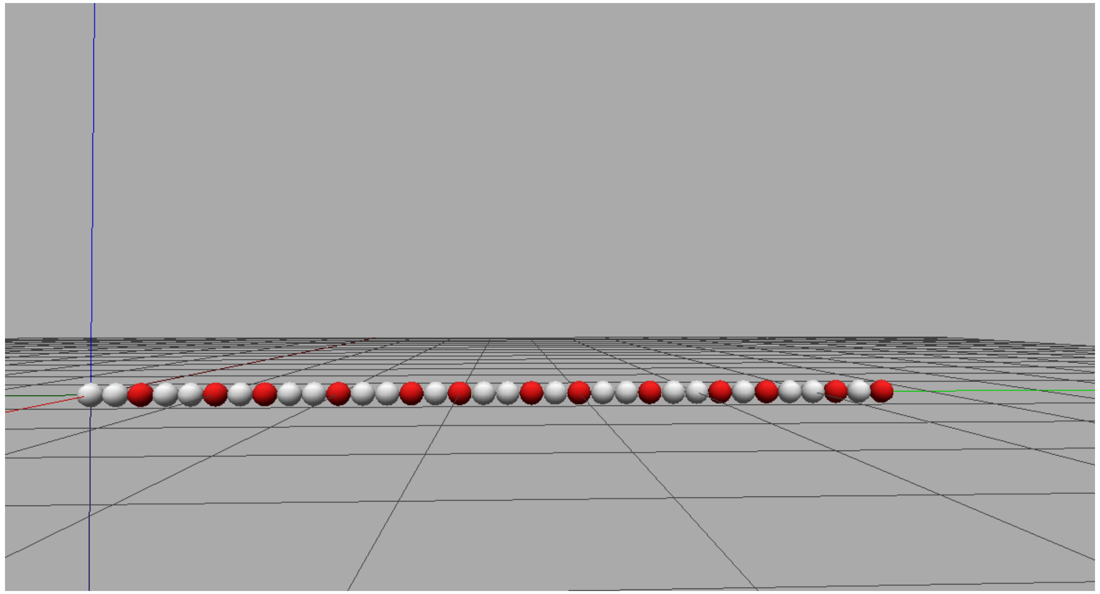
L-Py (interpretation)

- Drawing a cylinder with capital letter F:

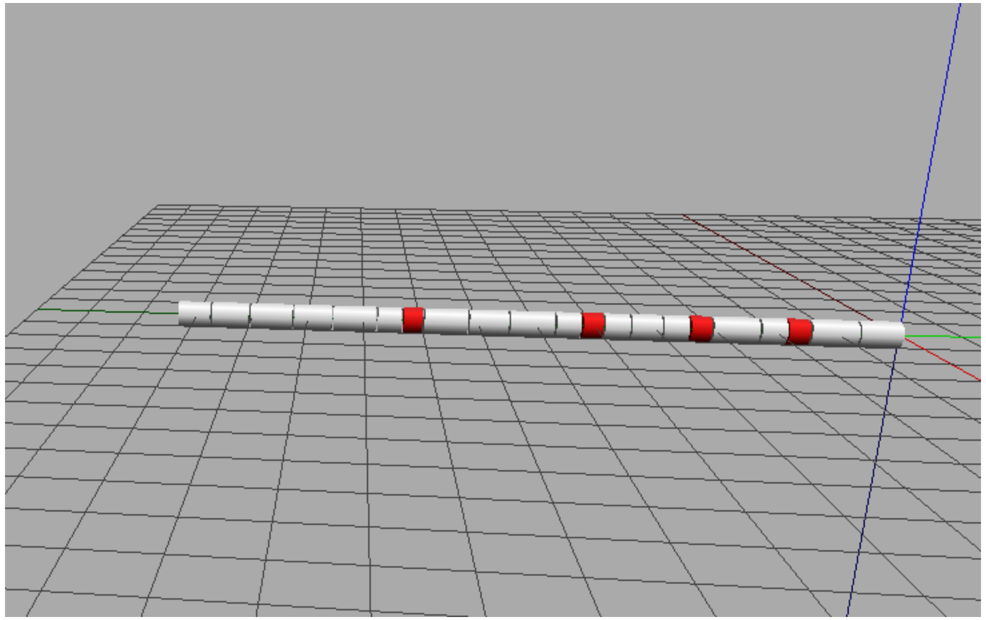
```
F(length)
```



1D growing filaments



Interpreted with spheres

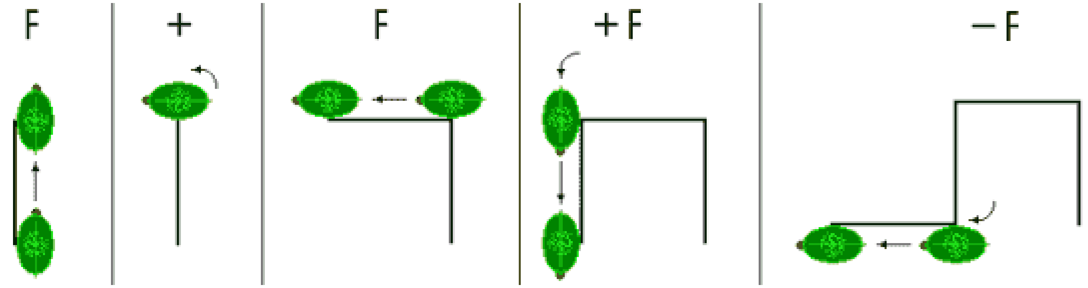


Interpreted with cylinders

Branching structures?

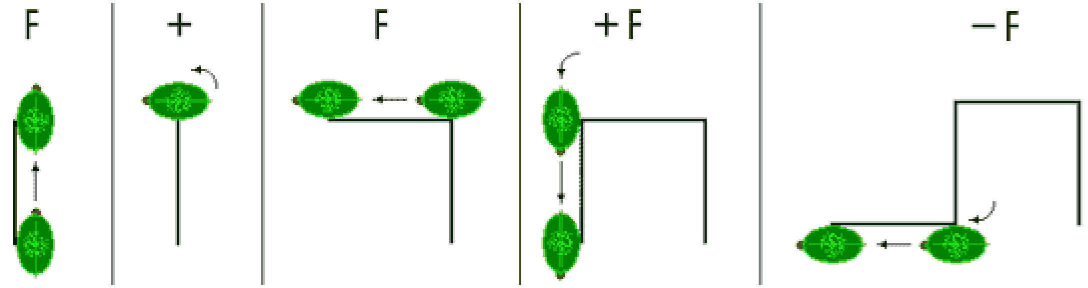
Turtle graphics

F+F+F-F

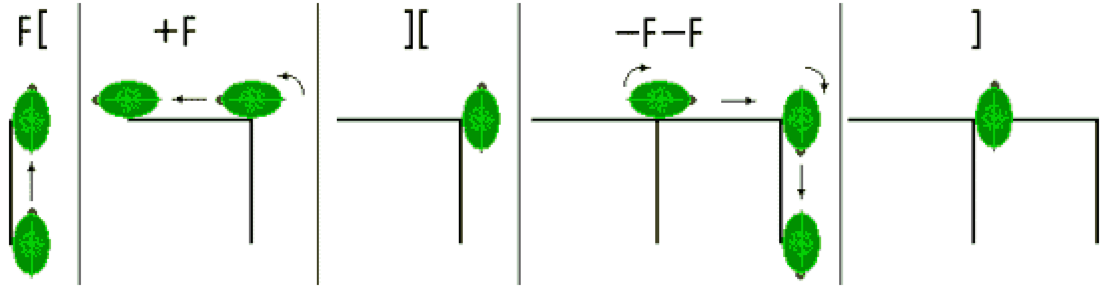


Turtle graphics

F+F+F-F



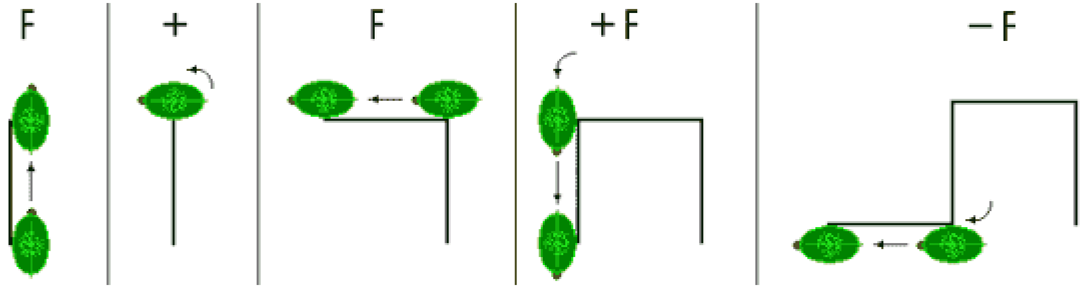
F[+F][-F-F]F



Turtle graphics

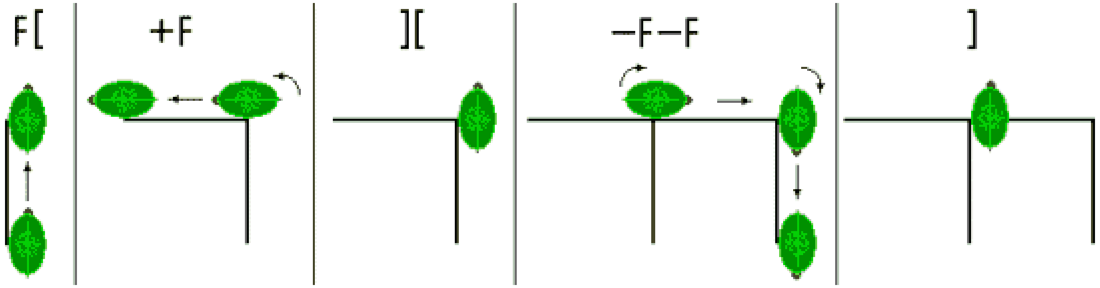
L-Py: F(1)+(90)...

F+F+F-F



L-Py: F(1)[+(90)F(1)]...

F[+F][-F-F]F

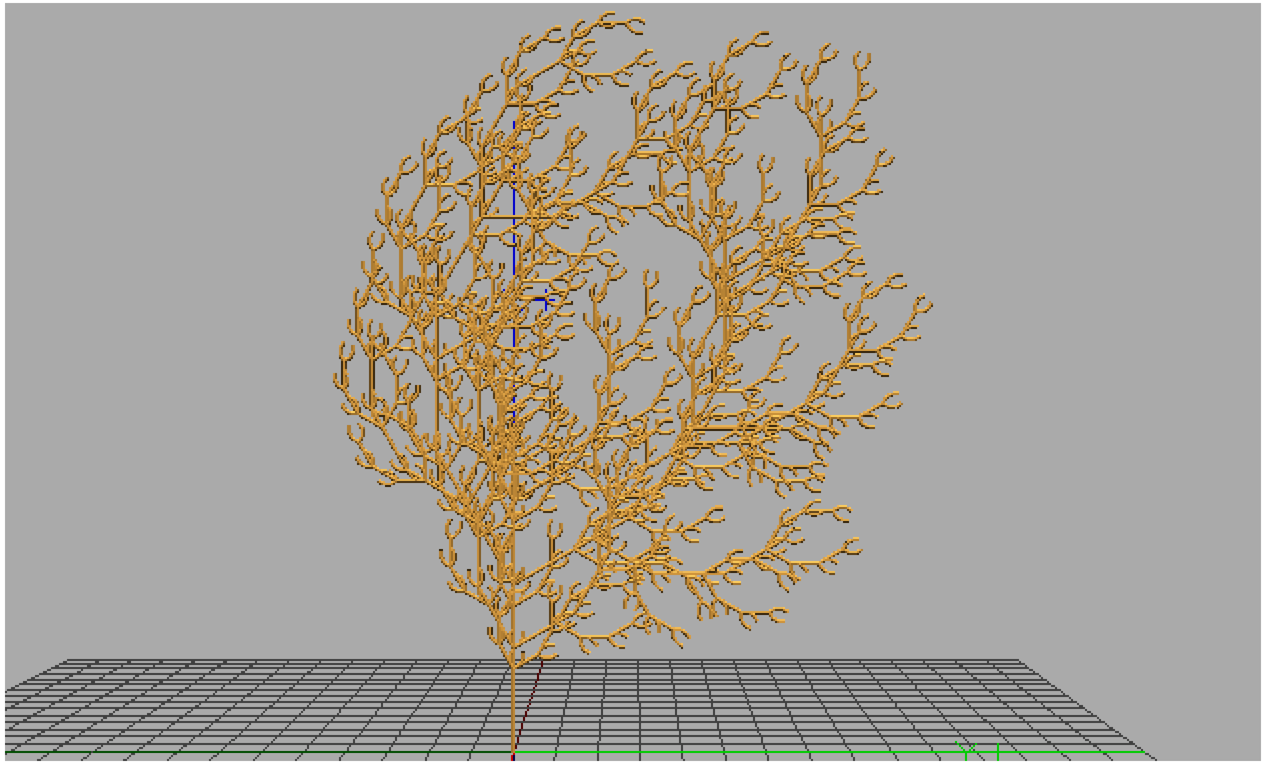


Branching structures 2D

$\omega: \quad F$

$p_1: \quad F \rightarrow FF[--F+F+F][+F-F-F]$

$\delta = 30°$



Stochastic L-System

L-Py:

import random

random.random() returns a number of the uniform distribution [0.0 –> 1.0]

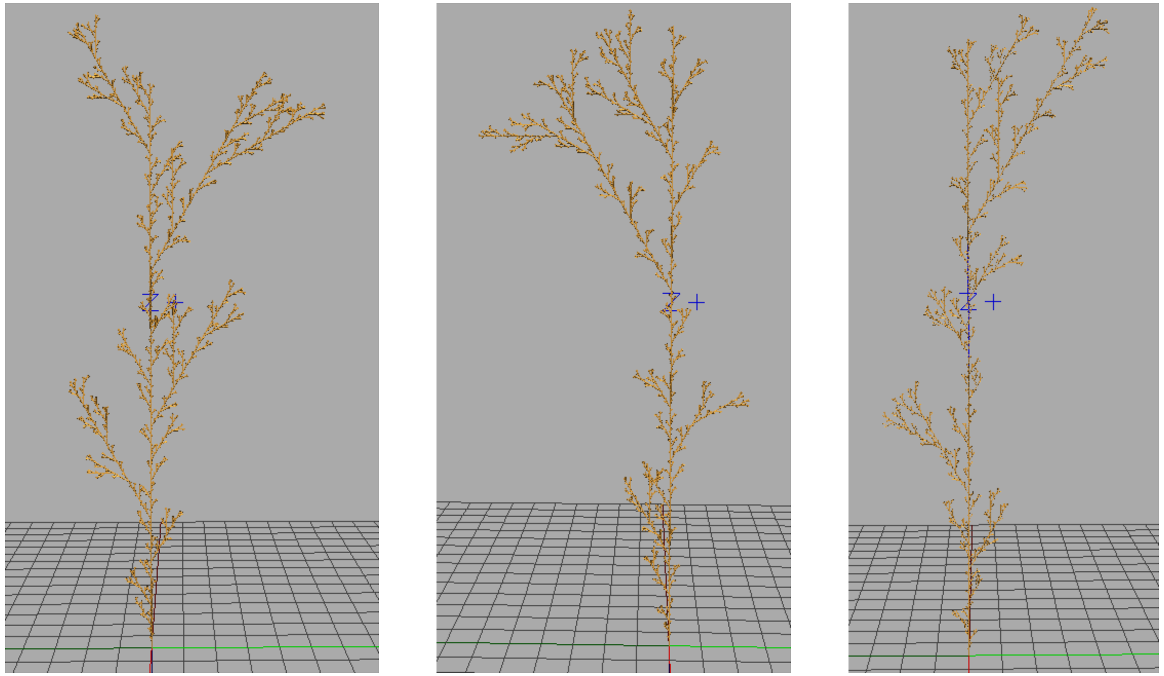
$\omega: \quad F$

$p_1: \quad F \xrightarrow{0.33} F[+F]F[-F]F$

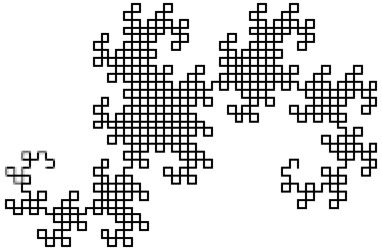
$p_2: \quad F \xrightarrow{0.33} F[+F]F$

$p_3: \quad F \xrightarrow{0.33} F[-F]F$

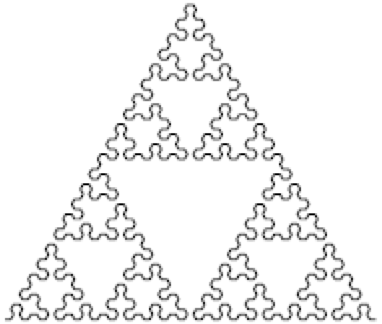
$\delta = 30°$



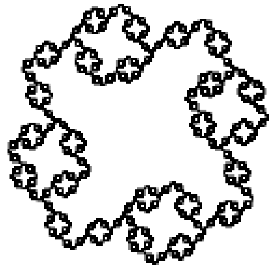
Fractals with L-Systems



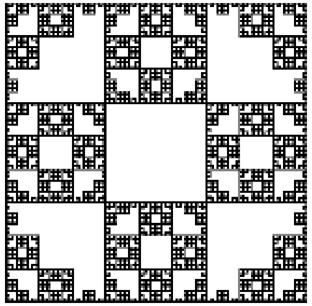
a $n=10$, $\delta=90°$
F_l
$F_l \to F_l + F_r +$
$F_r \to - F_l - F_r$



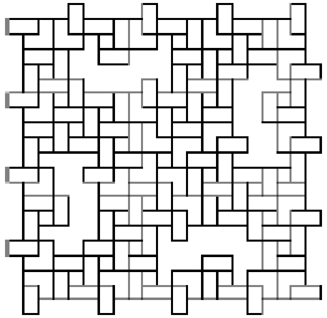
b $n=6$, $\delta=60°$
F_r
$F_l \to F_r + F_l + F_r$
$F_r \to F_l - F_r - F_l$



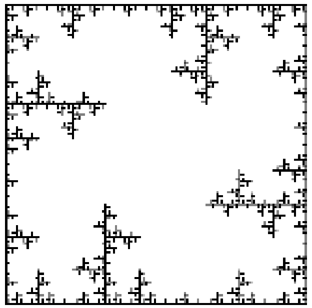
a $n = 4$, $\delta = 90°$
F-F-F-F
F → FF-F-F-F-F-F+F



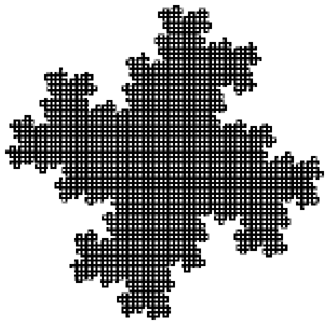
b $n = 4$, $\delta = 90°$
F-F-F-F
F → FF-F-F-F-FF



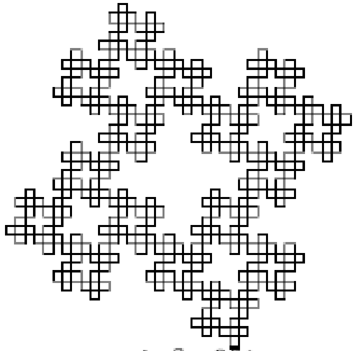
c $n = 3$, $\delta = 90°$
F-F-F-F
F → FF-F+F-F-FF



d $n = 4$, $\delta = 90°$
F-F-F-F
F → FF-F--F-F



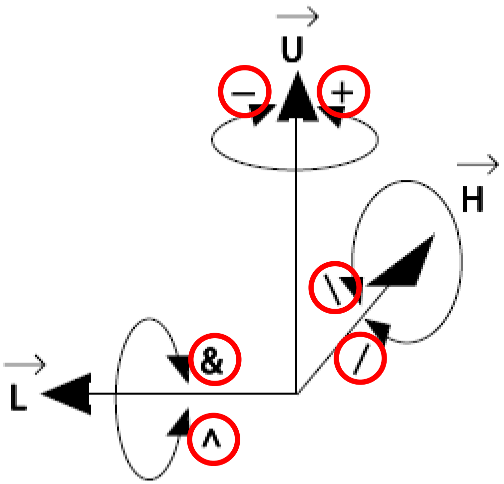
e $n = 5$, $\delta = 90°$
F-F-F-F
F → F-FF--F-F



f $n = 4$, $\delta = 90°$
F-F-F-F
F → F-F+F-F-F

Algorithmic beauty of plants. P. Prusinkiewicz 1992

L-System 3D



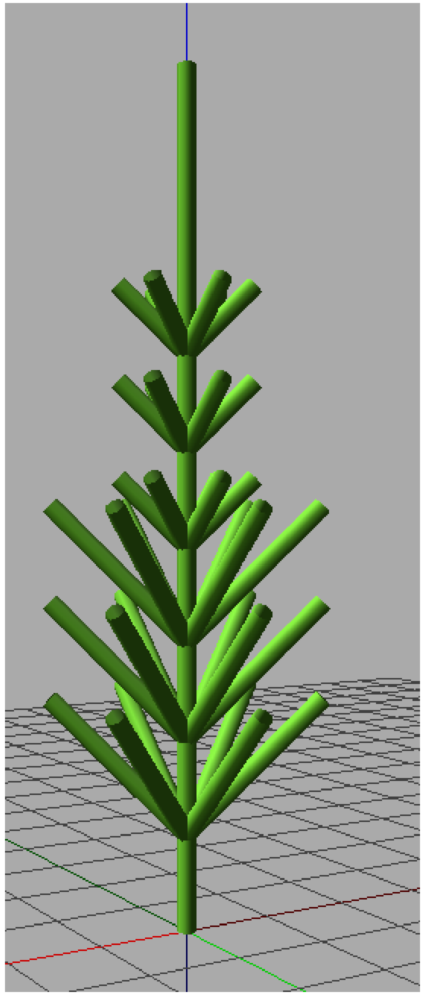
$$R_U(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_L(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

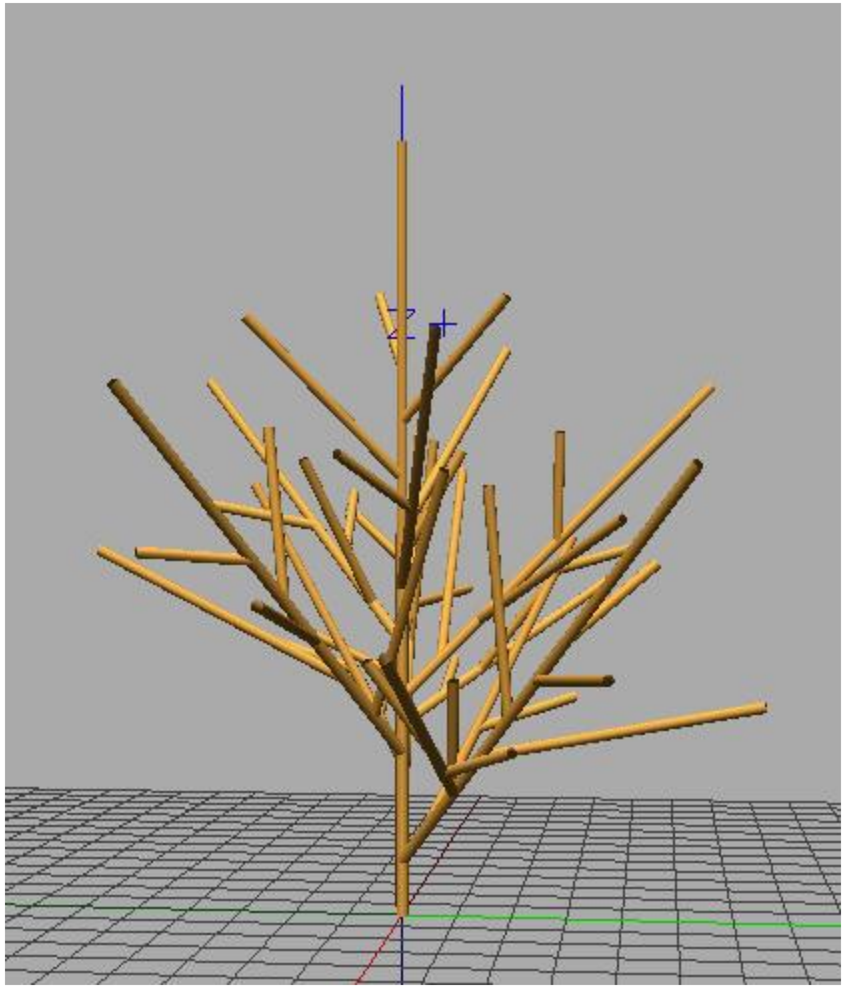
$$R_H(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\begin{bmatrix} \vec{H'} & \vec{L'} & \vec{U'} \end{bmatrix} = \begin{bmatrix} \vec{H} & \vec{L} & \vec{U} \end{bmatrix} \mathbf{R},$$

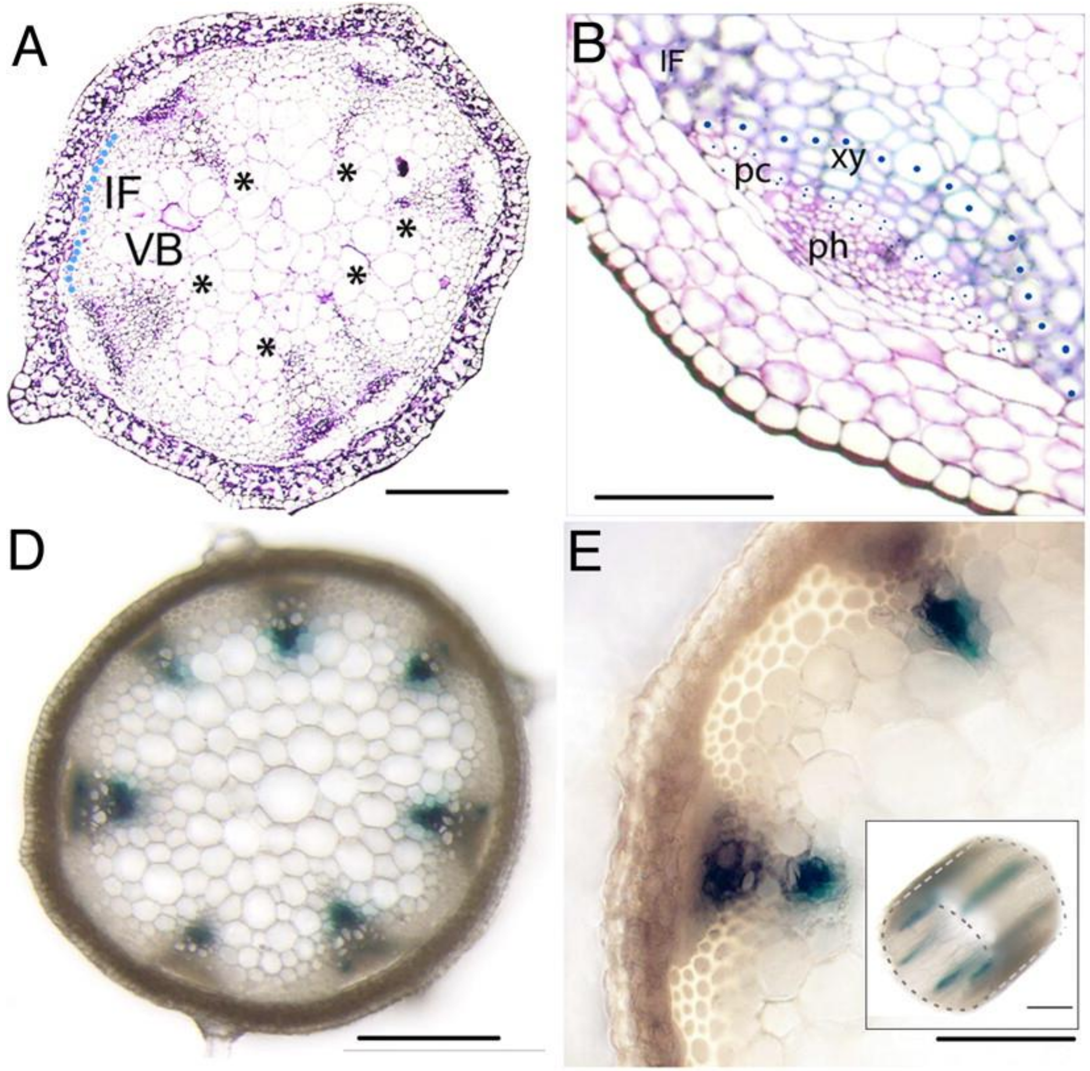
Equisetum Arvense



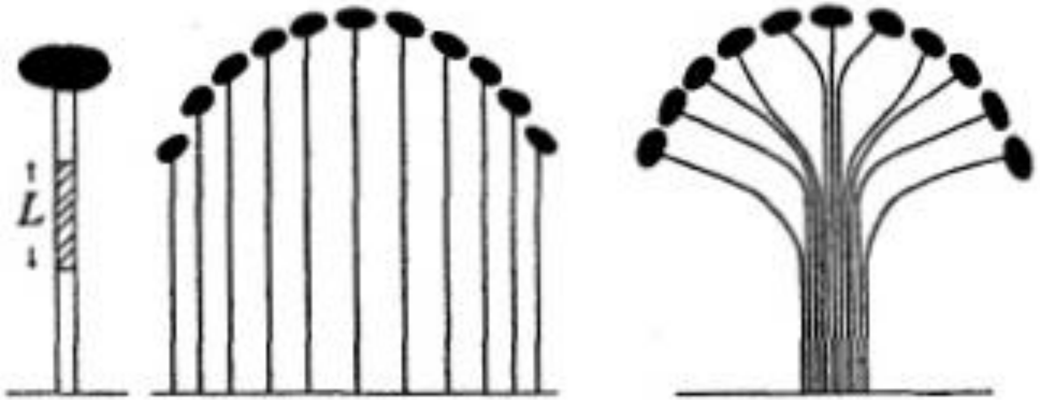
Modeling secondary growth



Cross-sections of plant stems

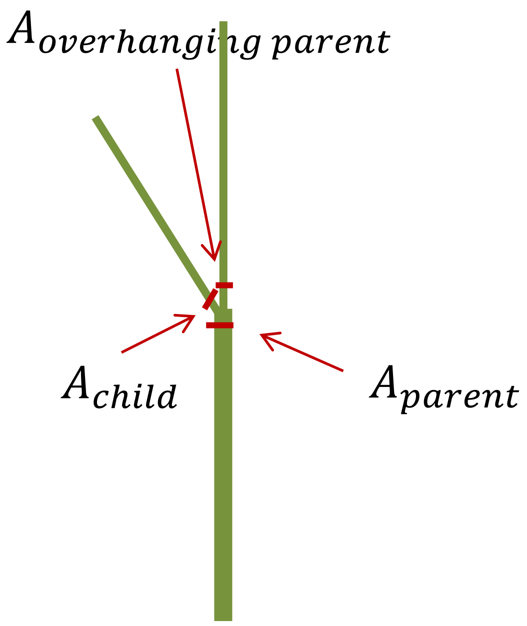


Pipe model

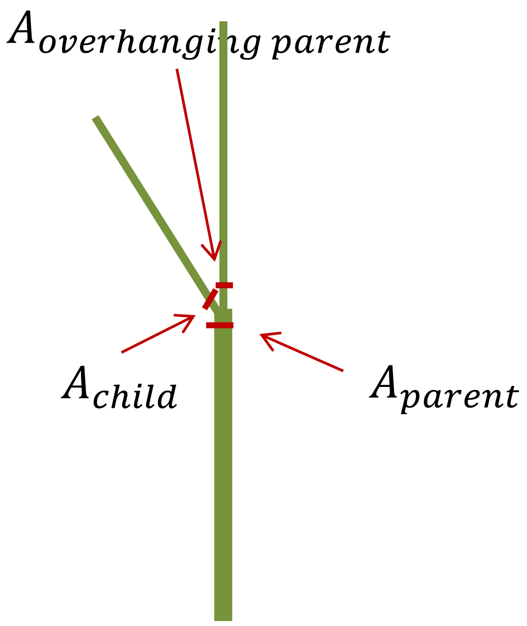


Shinozaki et al. 1964

Pipe model

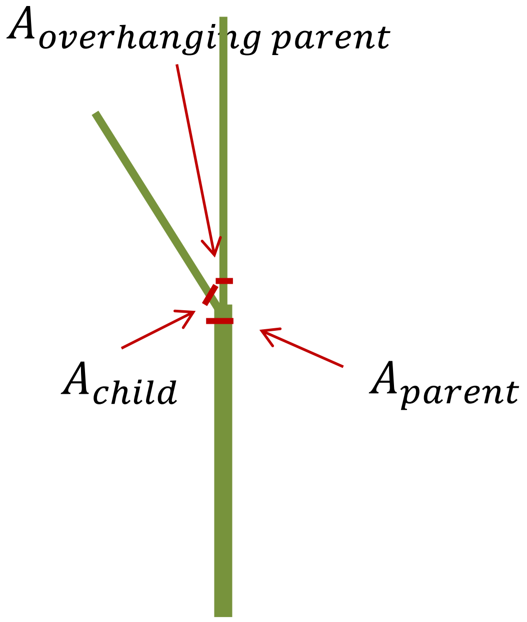


Pipe model



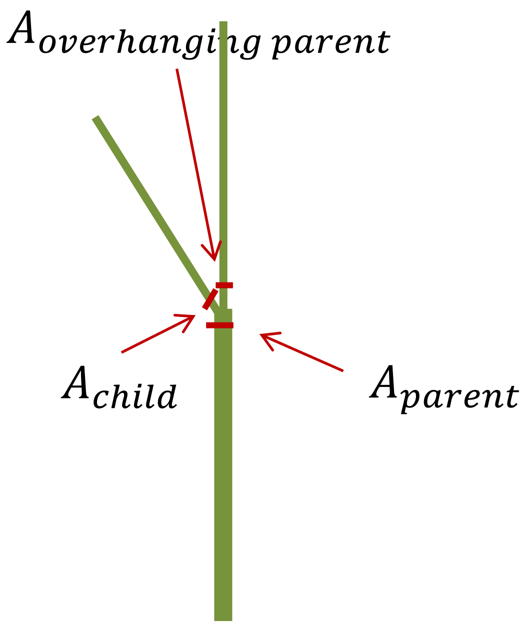
$$A_p = A_c + A_{op}$$

Pipe model



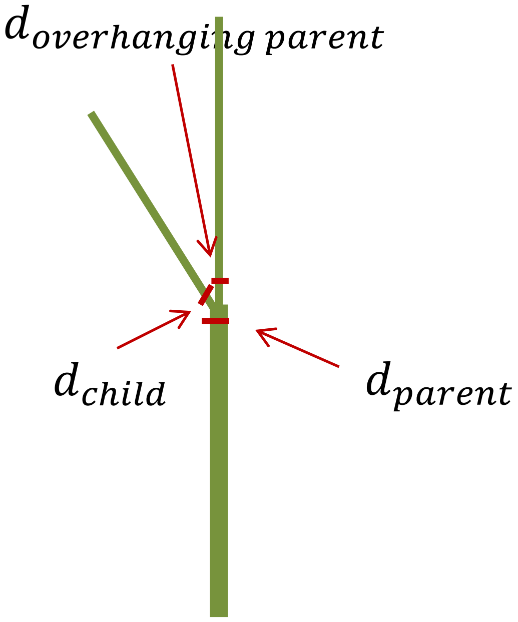
$$\pi r_p^2 = \pi r_c^2 + \pi r_{op}^2$$

Pipe model



$$r_p^2 = r_c^2 + r_{op}^2$$

Pipe model



$$d_p^n = d_c^n + d_{op}^n$$

$$n \in [1,3]$$

Context-sensitive L-Systems

$S < F \rightarrow E$ F becomes E, if it is before S in the string

$F > S \rightarrow E$ F becomes E, if it is after S in the string

$T < F > S \rightarrow E$ F becomes E, if it is before T and after S

Context-sensitive L-Systems

S **<** F → E F becomes E, if it is before S in the string

F **>** S → E F becomes E, if it is after S in the string

T **<** F **>** S → E F becomes E, if it is before T and after S

$$
\begin{aligned}
\omega &: \quad baaaaaaaa \\
p_1 &: \quad b < a \quad \rightarrow b \\
p_2 &: \quad\quad\;\; b \quad \rightarrow a
\end{aligned}
$$

Context-sensitive L-Systems

S **<** F → E F becomes E, if it is before S in the string

F **>** S → E F becomes E, if it is after S in the string

T **<** F **>** S → E F becomes E, if it is before T and after S

$baaaaaaaa$

$$\begin{aligned} \omega &: \quad baaaaaaaa \\ p_1 &: \quad b < a \quad \rightarrow b \\ p_2 &: \qquad\quad b \quad \rightarrow a \end{aligned}$$

Context-sensitive L-Systems

$S < F \rightarrow E$ F becomes E, if it is before S in the string

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$T < F > S \rightarrow E$ F becomes E, if it is before T and after S

$$
\begin{aligned}
\omega &: \quad baaaaaaaa \\
p_1 &: \quad b < a \quad \rightarrow b \\
p_2 &: \quad \phantom{b <{}} b \quad \rightarrow a
\end{aligned}
$$

$$
\begin{aligned}
&baaaaaaaa \\
&abaaaaaaa
\end{aligned}
$$

Context-sensitive L-Systems

S **<** F → E F becomes E, if it is before S in the string

F **>** S → E F becomes E, if it is after S in the string

T **<** F **>** S → E F becomes E, if it is before T and after S

$$
\begin{aligned}
\omega &: \quad baaaaaaaa \\
p_1 &: \quad b < a \quad \rightarrow b \\
p_2 &: \quad \phantom{b <{}} b \quad \rightarrow a
\end{aligned}
$$

$$
\begin{array}{l}
baaaaaaaa \\
abaaaaaaa \\
aabaaaaaa \\
aaabaaaaa \\
aaaabaaaa \\
\ldots
\end{array}
$$

Context-sensitive L-Systems []

S>[H]M

S[HI[JK]L]MNO

Context-sensitive L-Systems []

S>[H]M

S[HI[JK]L]MNO

L-Py: ignore: +-

Ignore specific symbols in the L-System string

Context-sensitive parametric L-Systems

$$A(x) < B(y) > C(z): \quad x + y + z > 10 \rightarrow E\left(\frac{x+y}{2}\right)F\left(\frac{y+z}{2}\right)$$

Context-sensitive parametric L-Systems

$$A(x) < B(y) > C(z): \quad x + y + z > 10 \rightarrow E\left(\frac{x+y}{2}\right)F\left(\frac{y+z}{2}\right)$$

$$\dots A(4)B(5)C(6)\dots$$

Context-sensitive parametric L-Systems

$$A(x) < B(y) > C(z): \quad x + y + z > 10 \rightarrow E\left(\frac{x+y}{2}\right)F\left(\frac{y+z}{2}\right)$$

$$\ldots A(4)B(5)C(6) \ldots$$

$$4 + 5 + 6 > 10$$

Context-sensitive parametric L-Systems

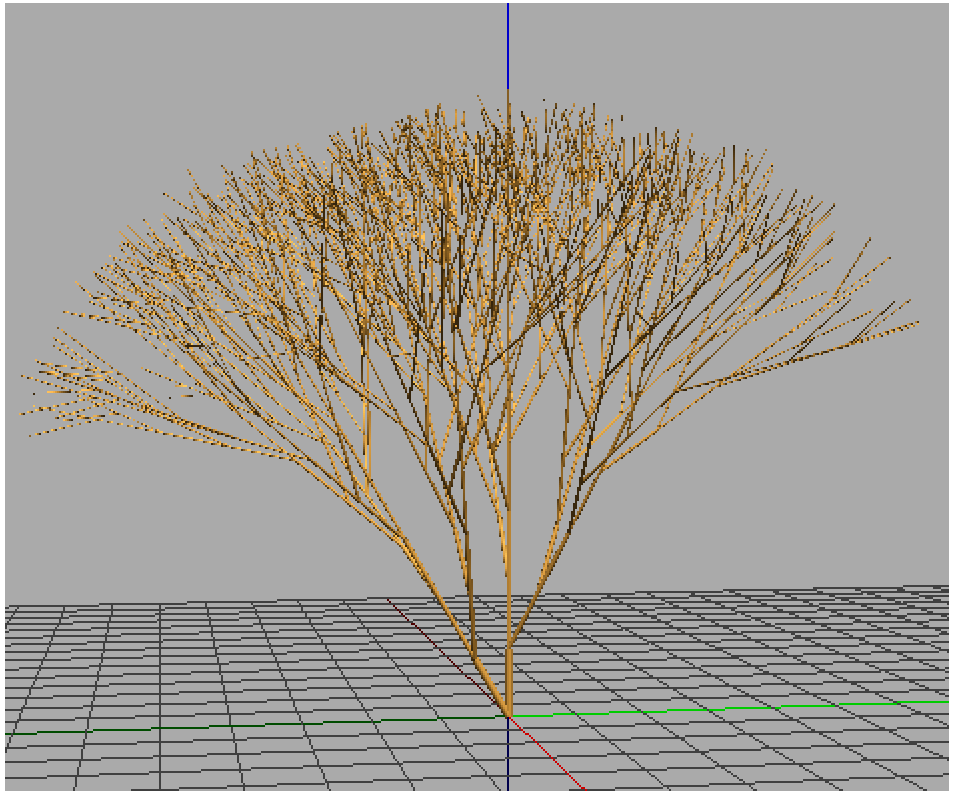
$$A(x) < B(y) > C(z): \quad x + y + z > 10 \rightarrow E\left(\frac{x+y}{2}\right)F\left(\frac{y+z}{2}\right)$$

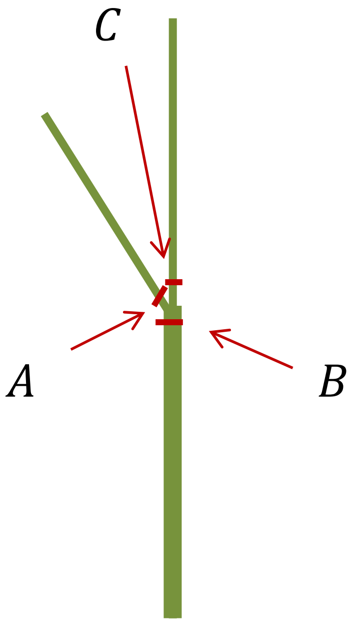
$$\ldots A(4)B(5)C(6)\ldots$$

$$4 + 5 + 6 > 10$$

$$B(5) \rightarrow E(4.5)F(5.5)$$

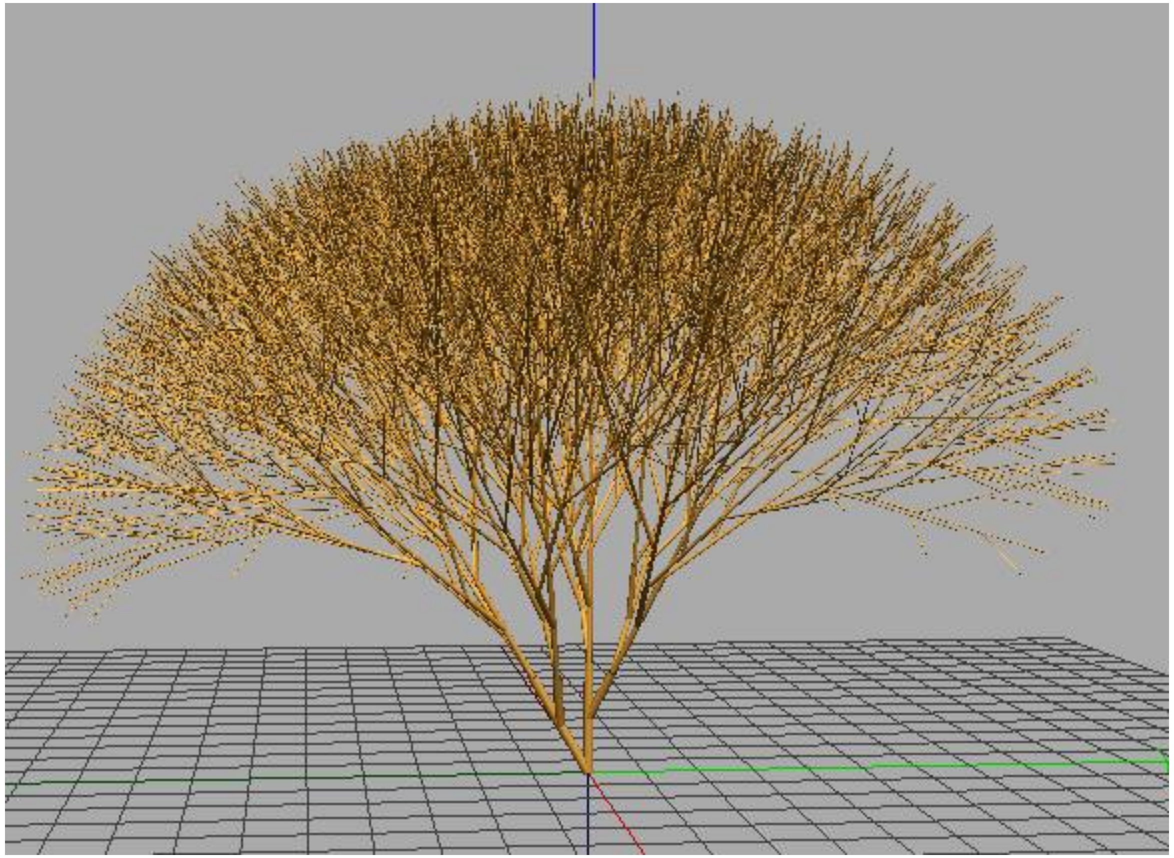
Context-sensitive pipe model



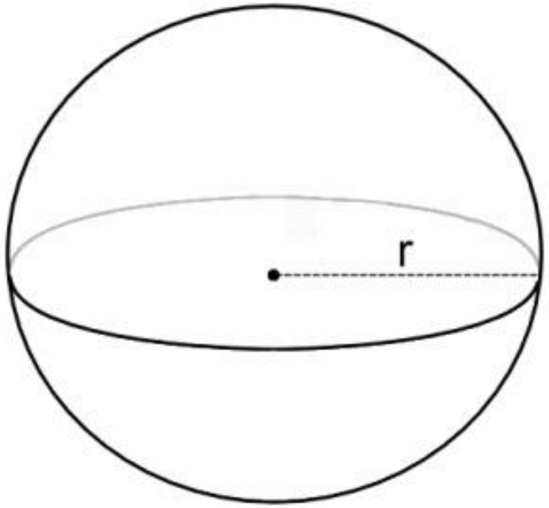


$$B(d_c) > [A(d_p)]C(d_{op}) \rightarrow d_p^n = d_c^n + d_{op}^n$$

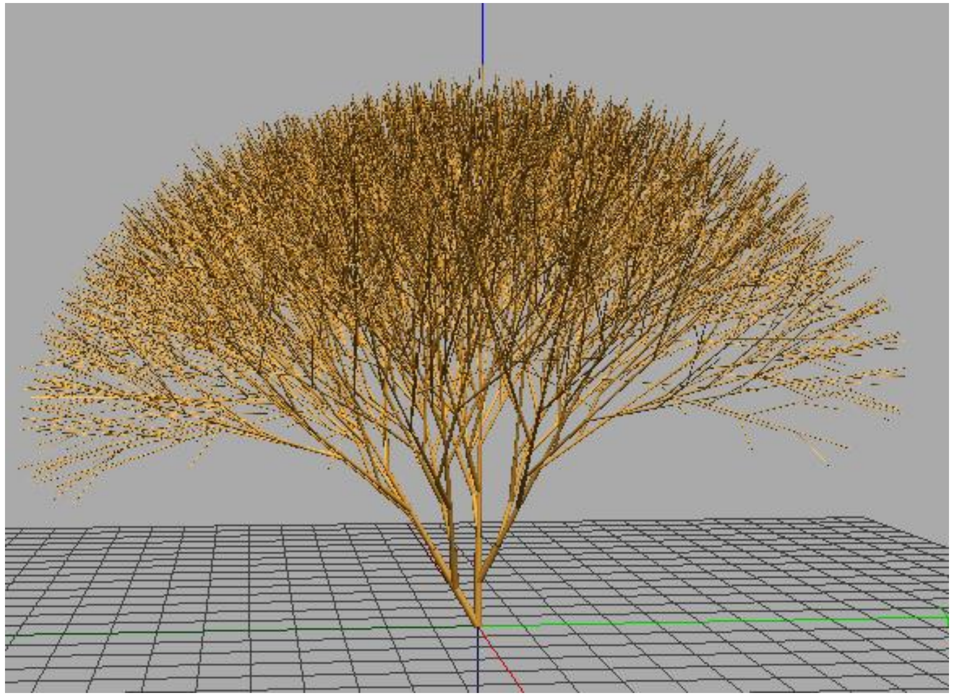
Dense branching structures unrealistic



Constraints of space

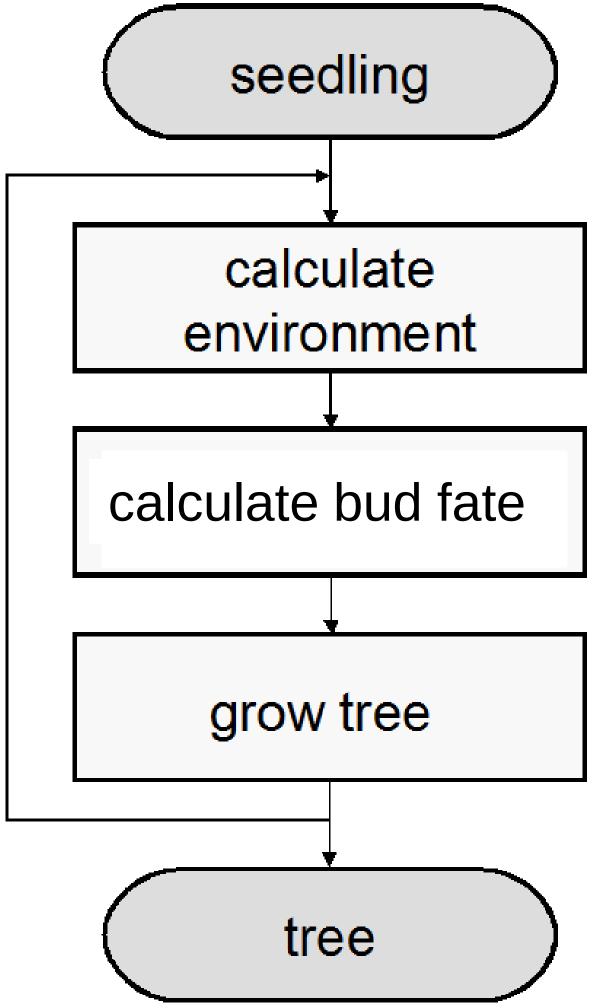


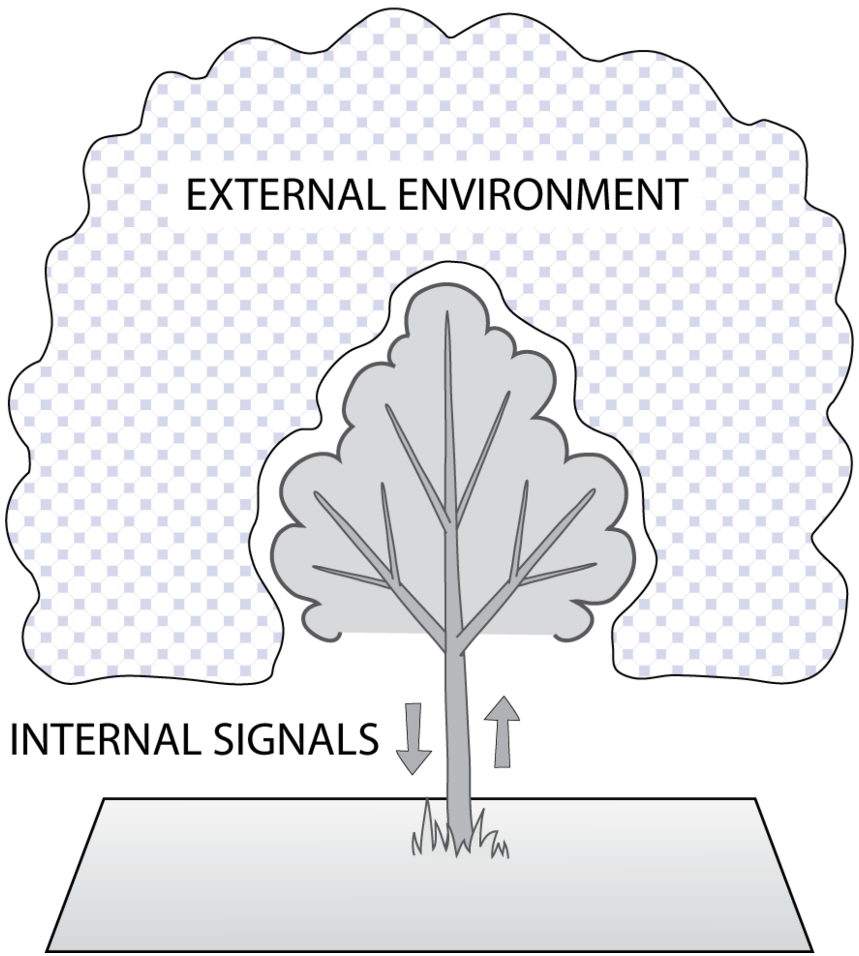
Sphere volume ~ r^3

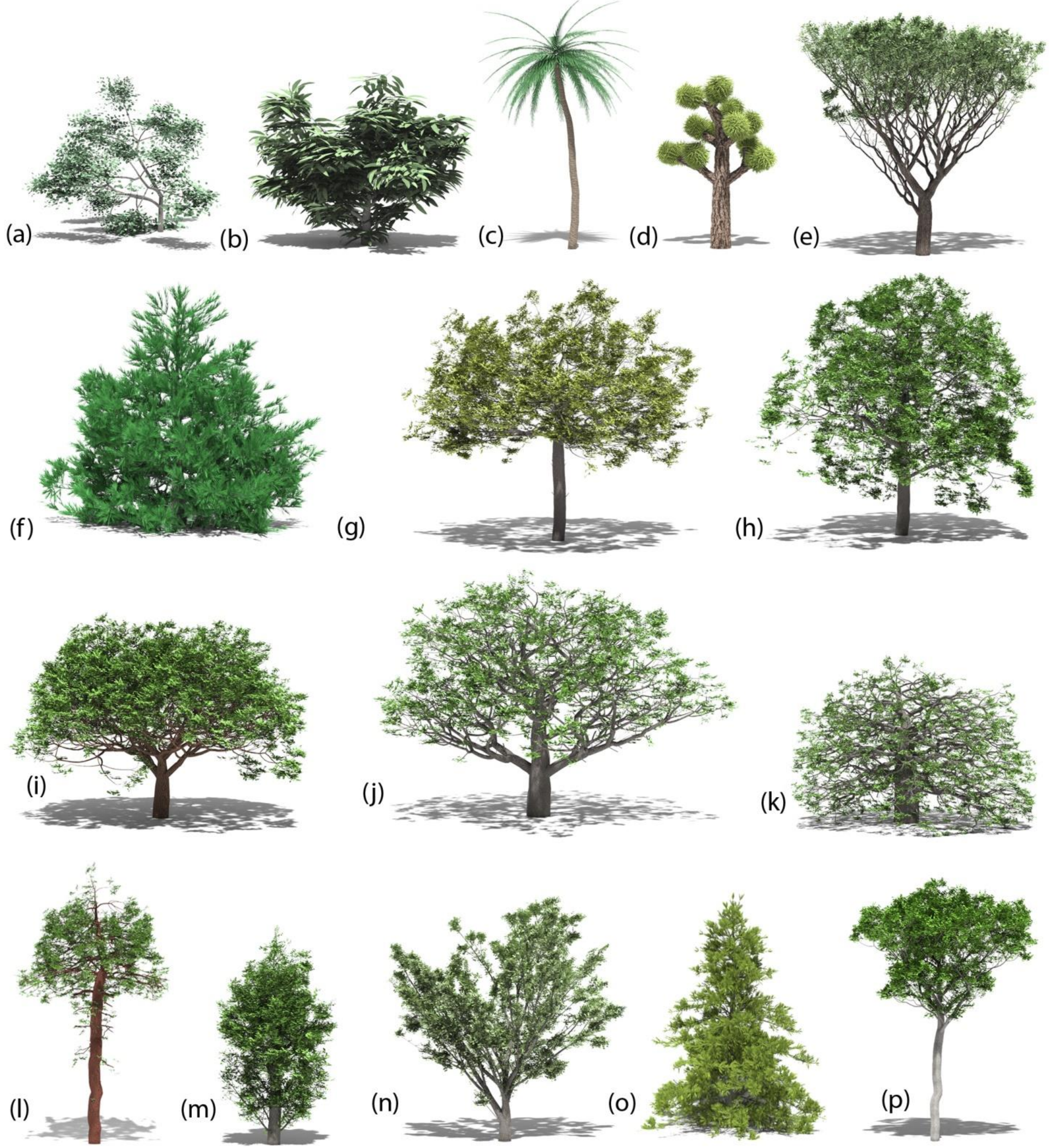


Number of branches ~ 2^r

Simulation Overview






(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)
(m)
(n)
(o)
(p)

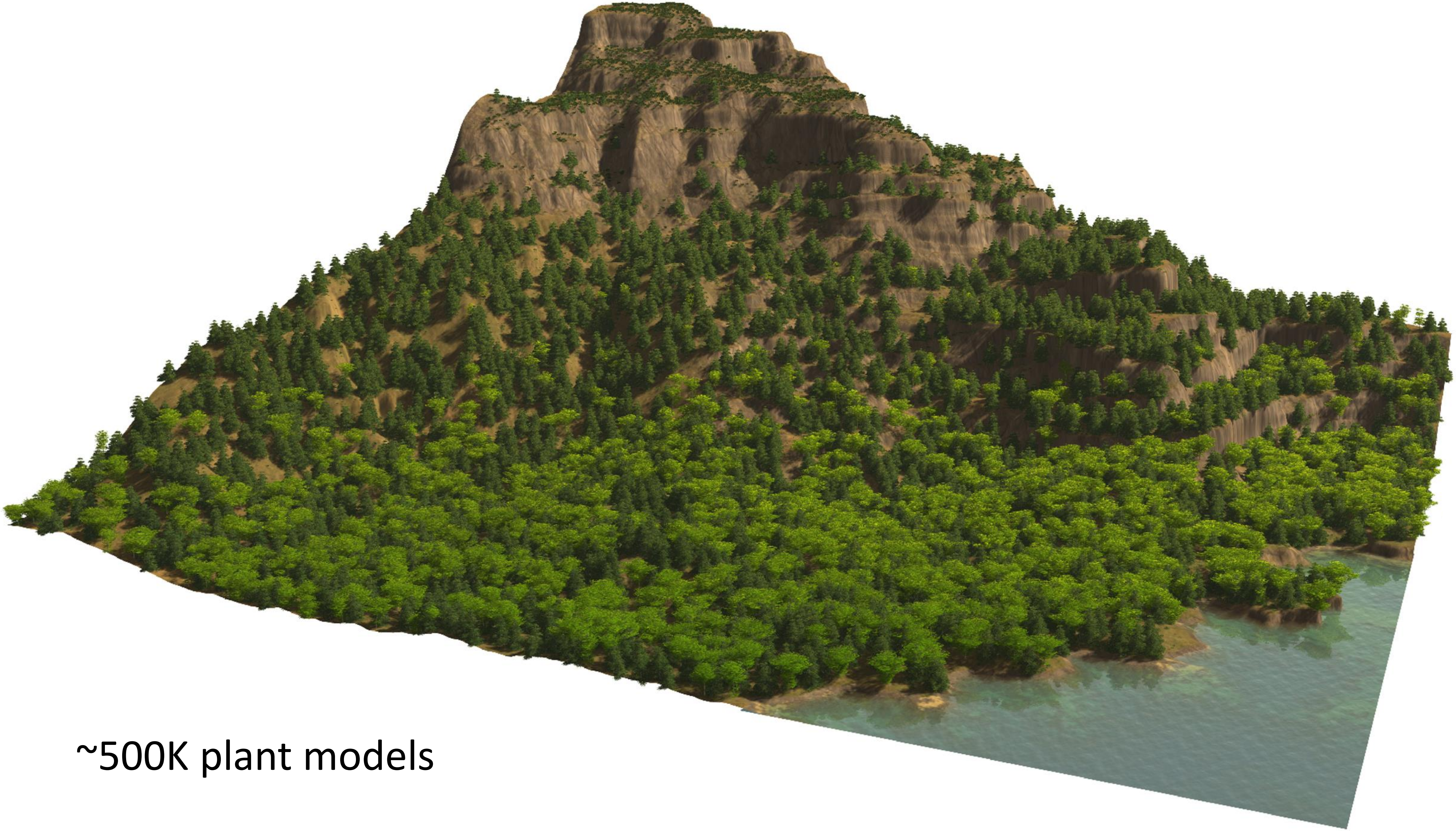
Comparison to real trees

Tabebuia rosea (Model of Leeuwenberg)

Phellodendron chinense (Model of Scarrone)

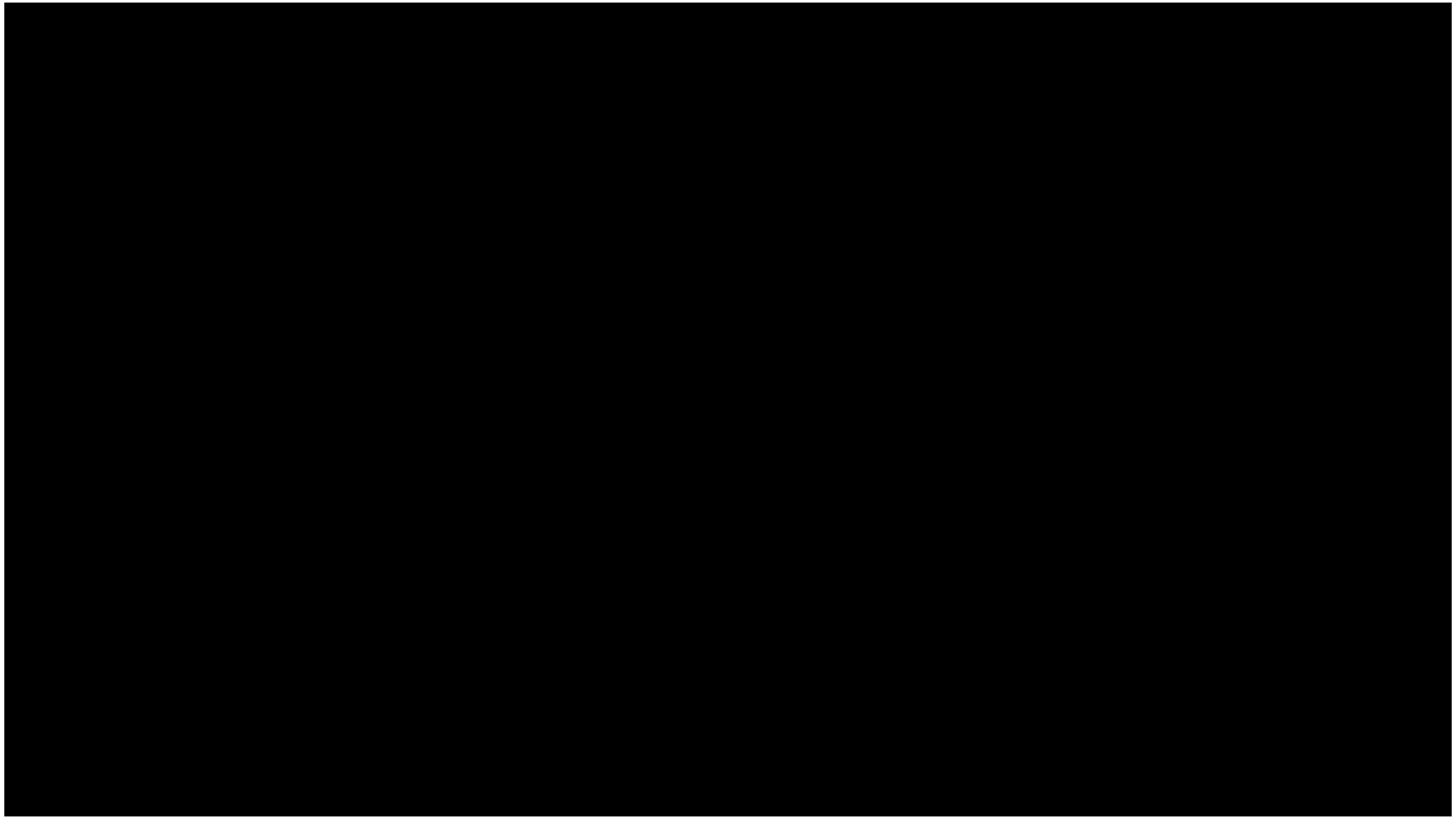
Delonix regia (Model of Troll)

Sequoia sempervirens (Model of Massart)



~500K plant models

PFE Project: flower development



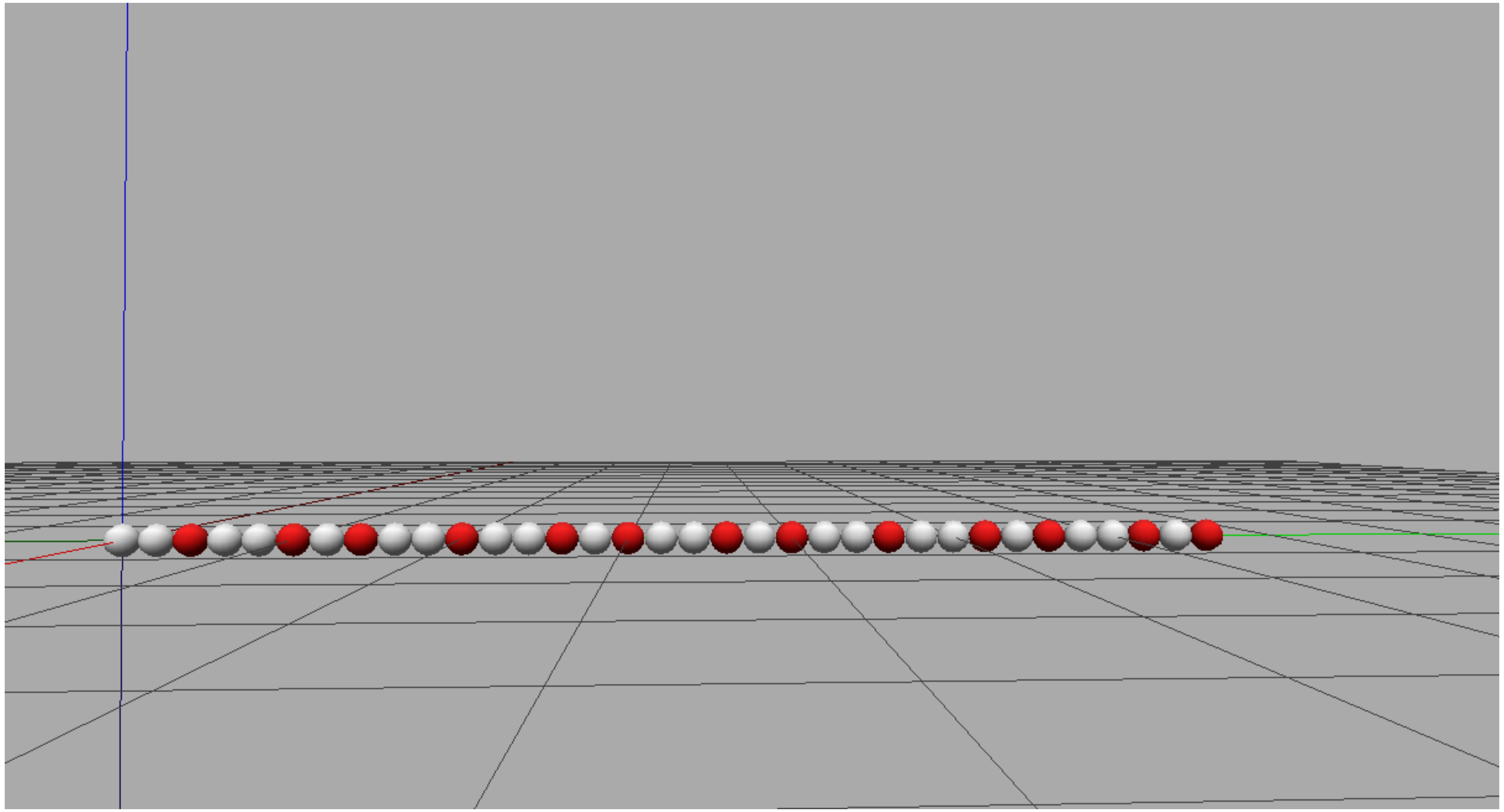
PFE Project: flower development

- Model and quantify the growth of a flower (alternatively your own idea, e.g. tree or aquatic plant growth)
- Implement it using L-Py and quantify the development using statistical methods provided by numpy and scipy
- Send me a pdf presentation with the source code before June 24th by e-mail or a youtube link of the presentation if you can't make it to the exam
- Present your results (24th June) in max. 5 minutes.
  - Explain which biological pattern you tried to model, give a brief description of the L-System used and show your results.
- Same groups as the diffusion presentations

PFE Project evaluation

- **~1/3**: *Complexity* of the L-System (context-sensitive, pipe model, parametric, differential equations)
- **~1/3 pts**: *Aesthetic appeal* (use L-Py swept surfaces, animated development with L-Py functions)
- **~1/3 pts**: *Statistical evaluation* that highlights key features of the synthetic development such as distributions of leaves, flowers and stem segments (using histograms, principal component analyses, regressions, testing of hypotheses, etc.)

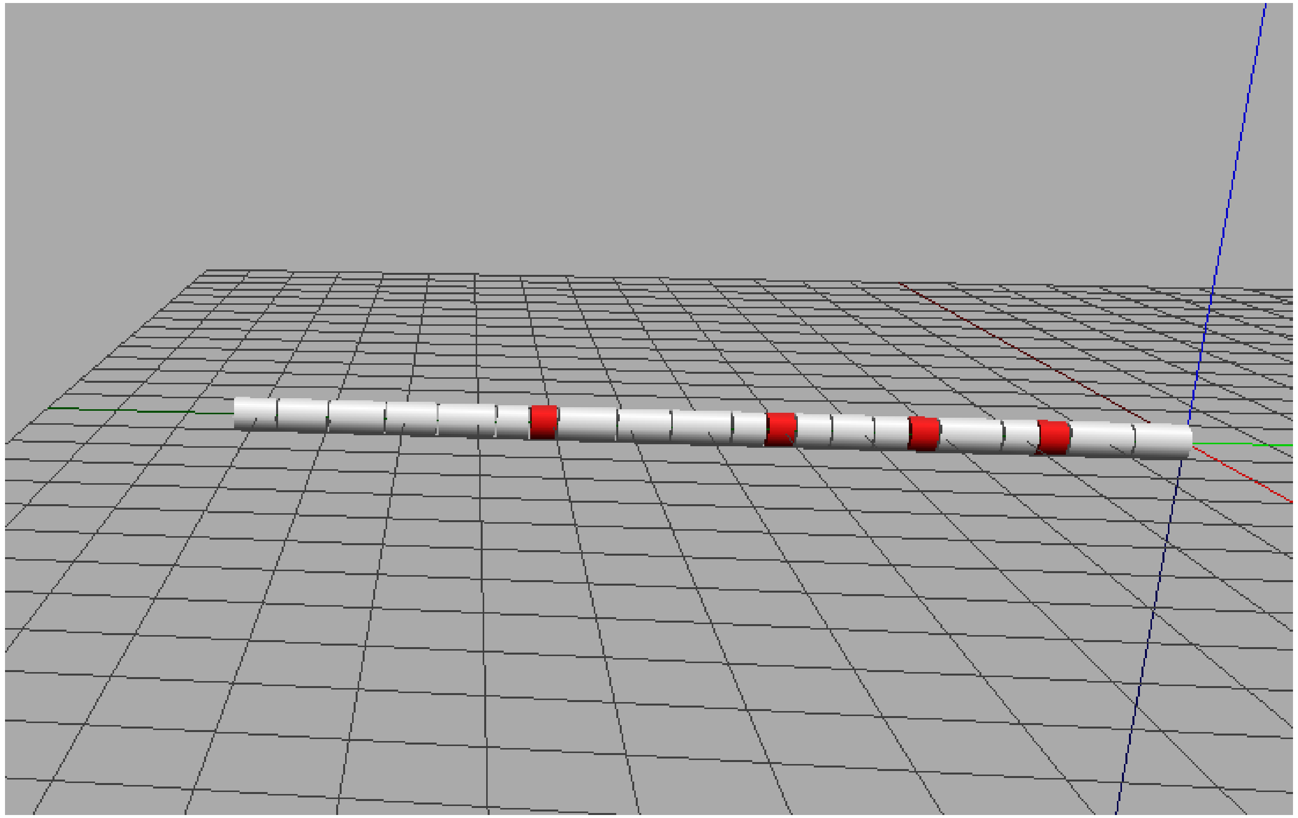
Exercise: Create an anabaena model with L-Py (slides 32-39)



Anabaena Catenula: Model 2

- The previous model does not take into account the size of cells
- In wet lab experiments it was observed that cells A divide every 15 hours and smaller cells B differentiate into cells of type A after 3 hours
- Express the gradual lengthening of cells A over the time period of 15 hours by using a **parametric L-System**
- A derivation step in the L-System can be set to **3 hours** (also called a plastochron)

Exercise: implement Model 2 with L-Py

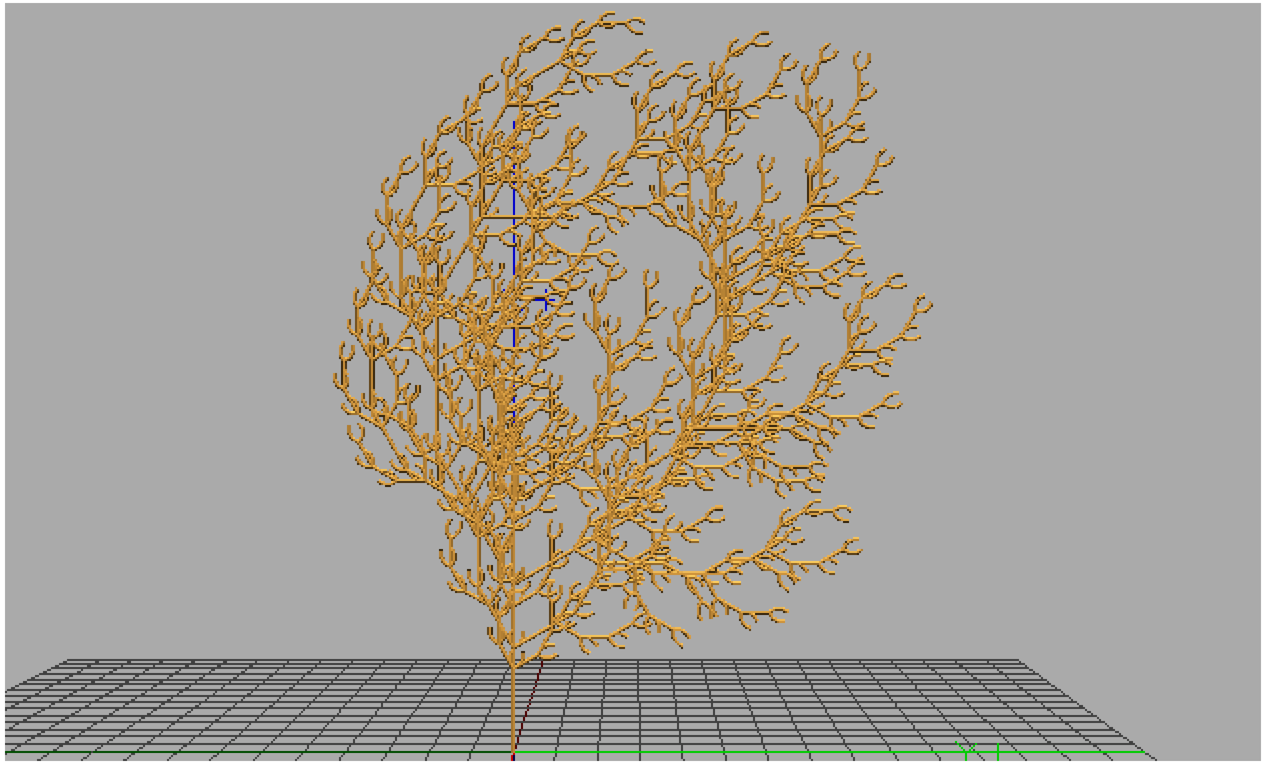


Exercise

$\omega: \quad F$

$p_1: \quad F \rightarrow FF[--F+F+F][+F-F-F]$

$\delta = 30°$



Exercise

L-Py:

import random

random.random() returns a number from [0,1]

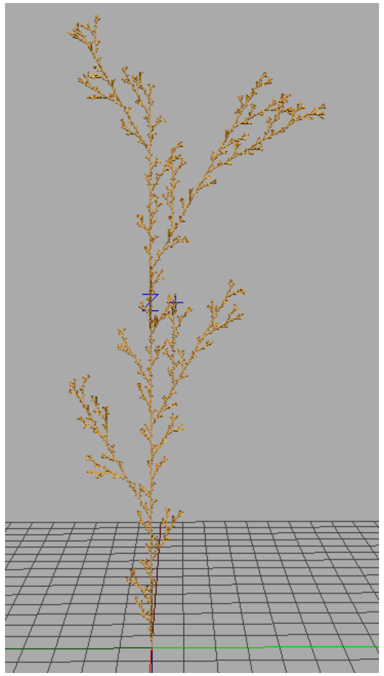
$\omega: \quad F$

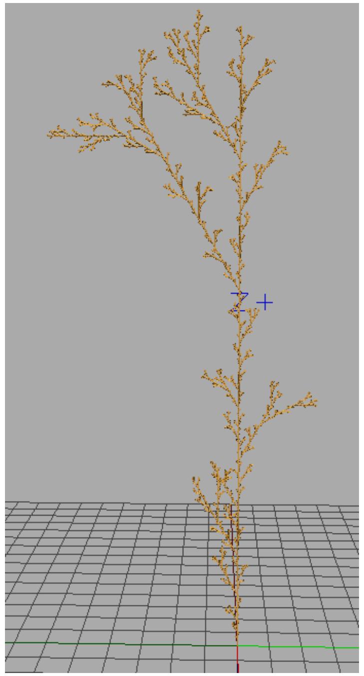
$p_1: \quad F \xrightarrow{0.33} F[+F]F[-F]F$

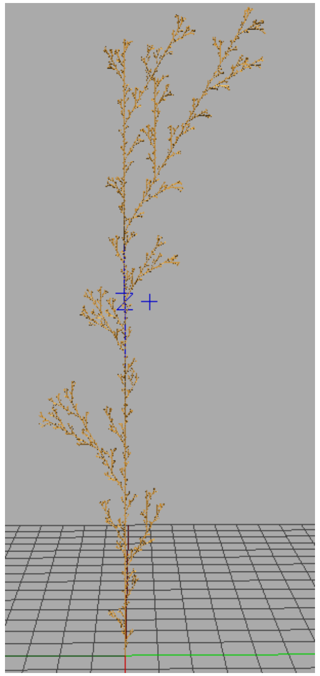
$p_2: \quad F \xrightarrow{0.33} F[+F]F$

$p_3: \quad F \xrightarrow{0.33} F[-F]F$

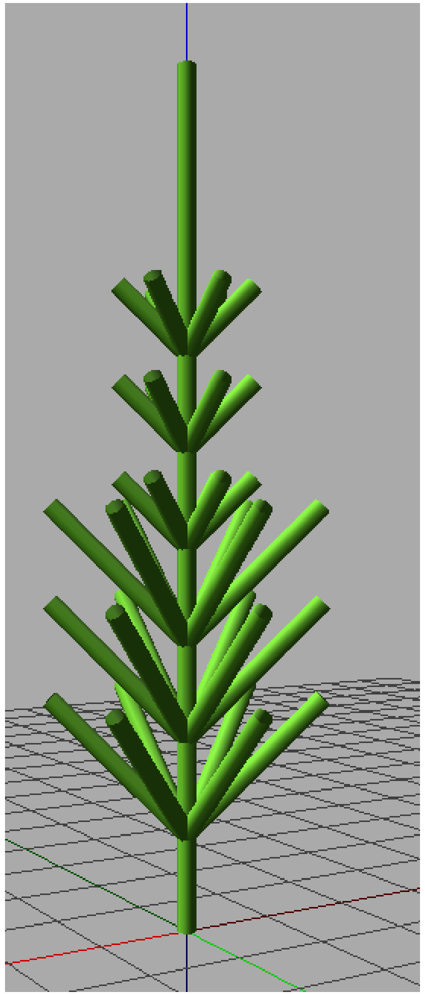
$\delta = 30°$







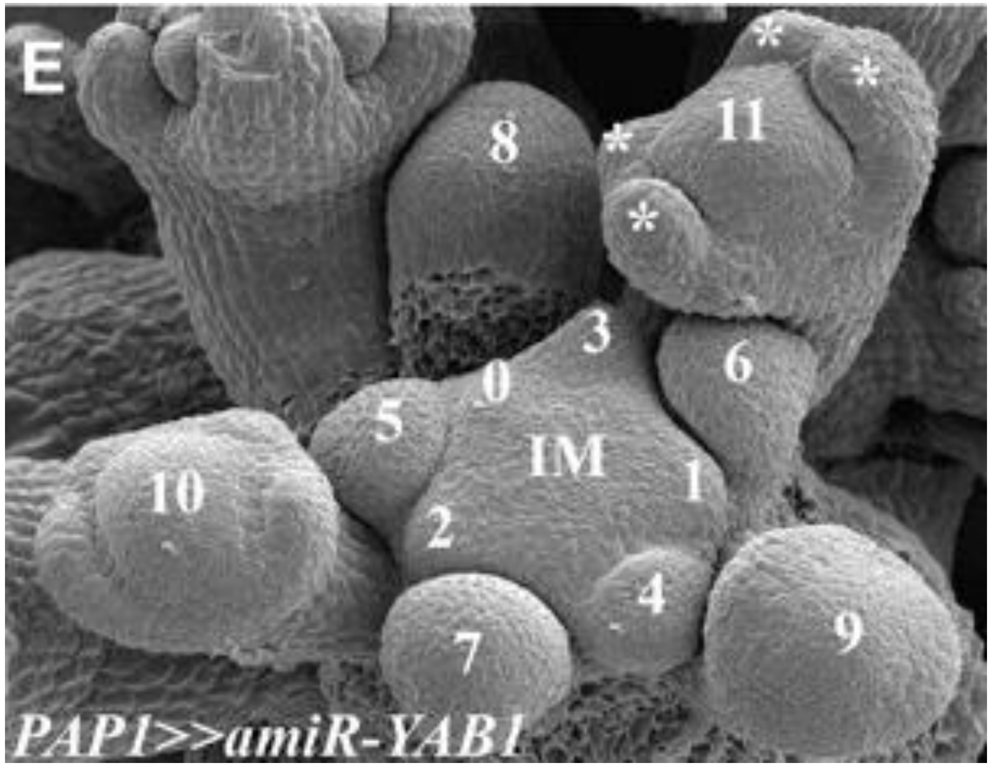
Exercise



Phyllotaxy

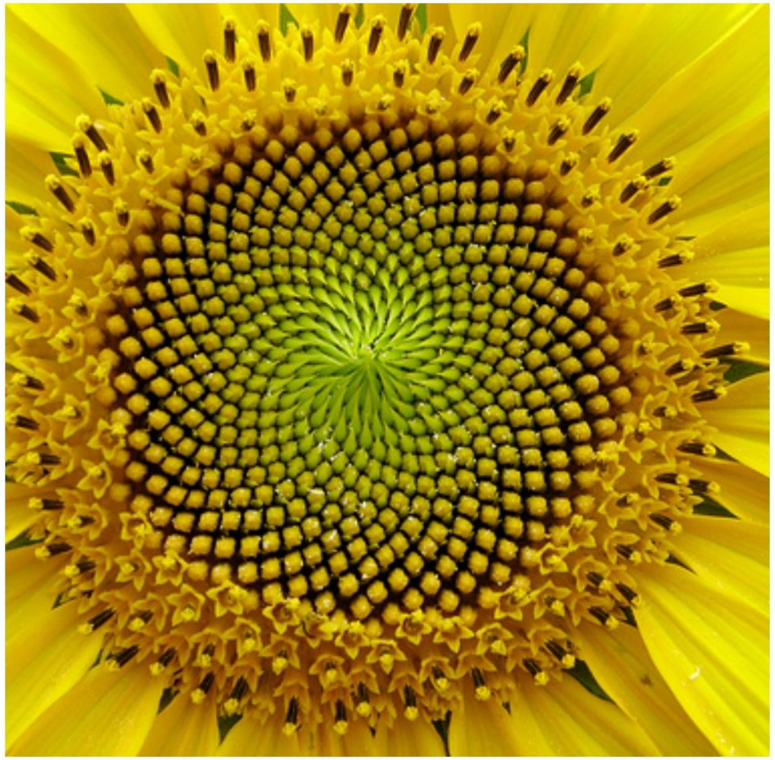
  

Apical meristem



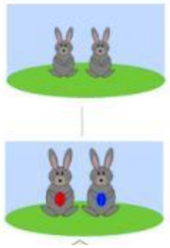
http://www.plantcell.org/content/20/5/1217/F5.expansion

How to mathematically model phyllotaxy?

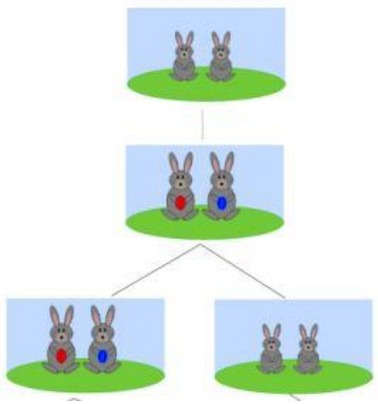


Fibonacci numbers

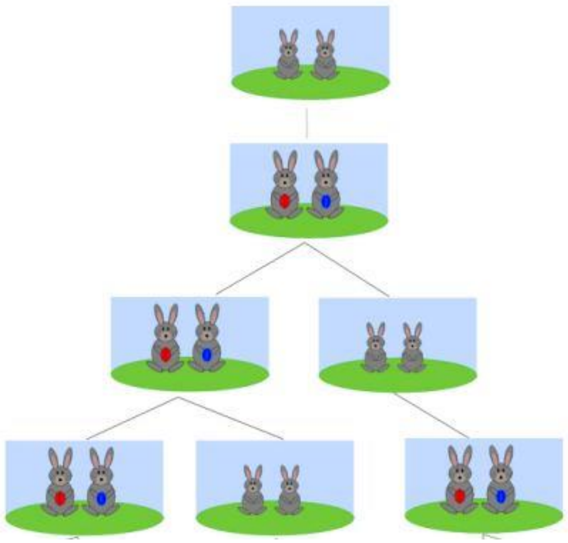
Fibonacci numbers



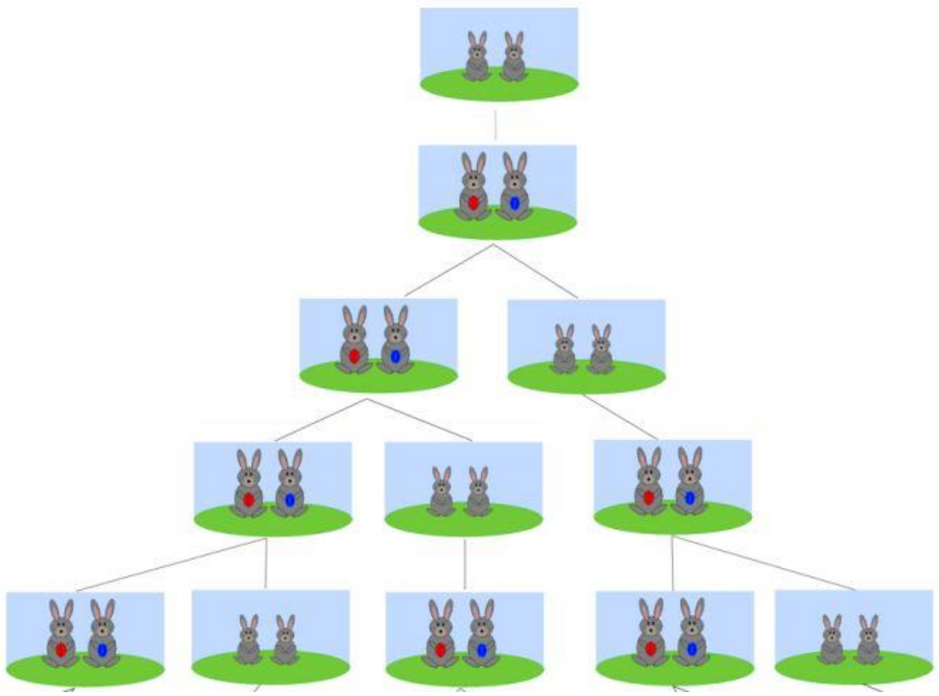
Fibonacci numbers



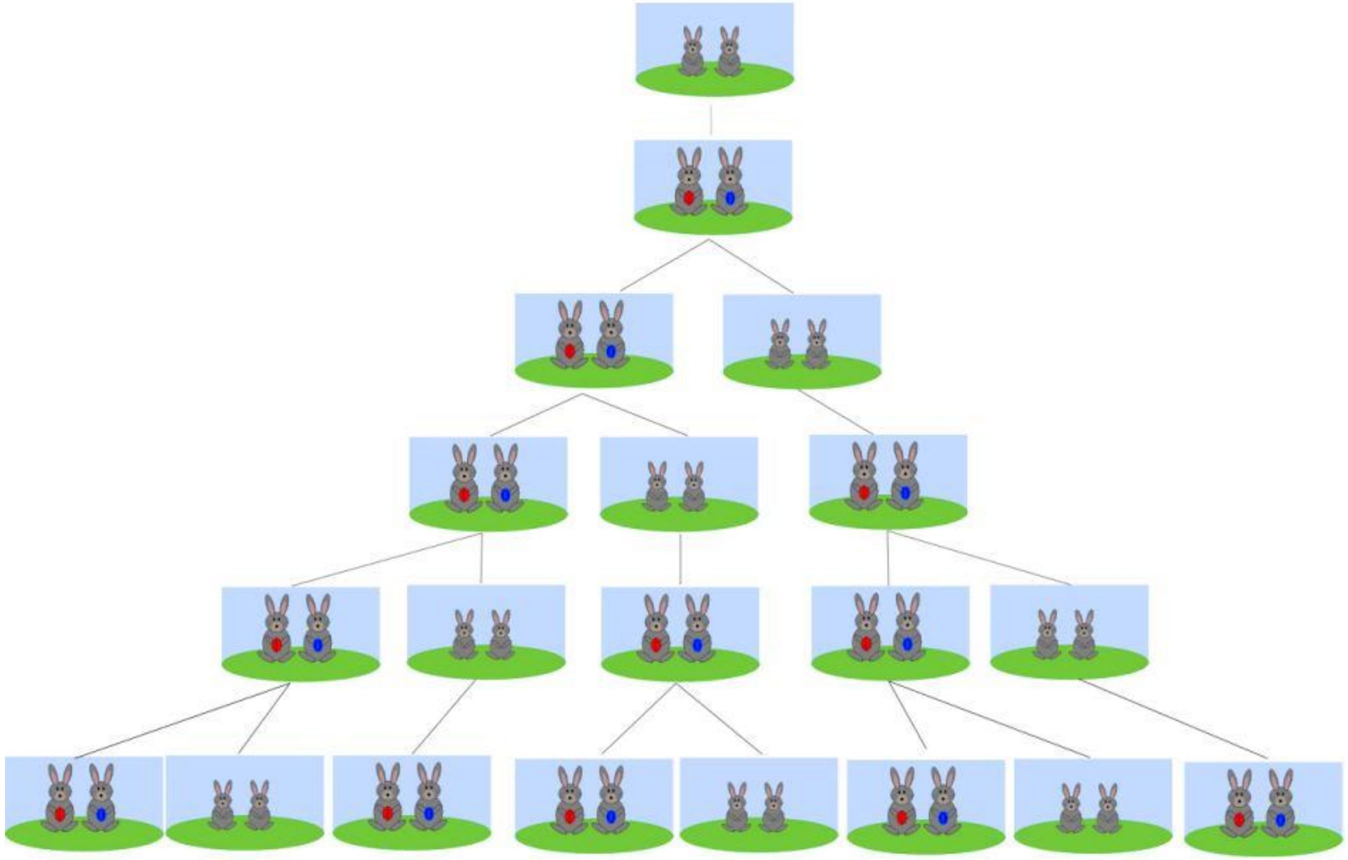
Fibonacci numbers



Fibonacci numbers



Fibonacci numbers



Annual rabbit population growth

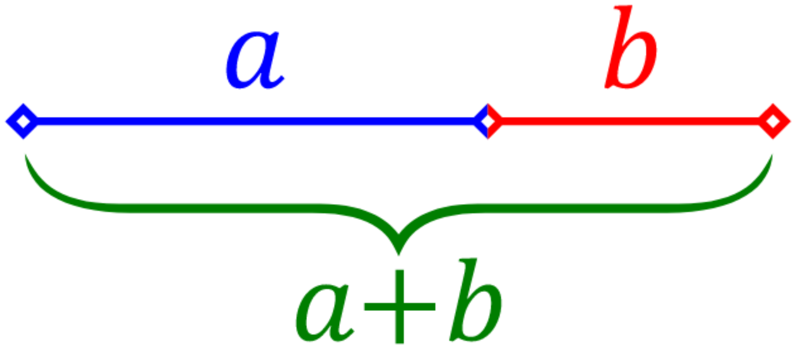
month	J	F	M	A	M	J	J	A	S	O	N	D	J
juvenile	1	0	1	1	2	3	5	8	13	21	34	55	89
adult	0	1	1	2	3	5	8	13	21	34	55	89	144
total	1	1	2	3	5	8	13	21	34	55	89	144	233

Annual rabbit population growth

month	J	F	M	A	M	J	J	A	S	O	N	D	J
juvenile	1	0	1	1	2	3	5	8	13	21	34	55	89
adult	0	1	1	2	3	5	8	13	21	34	55	89	144
total	1	1	2	3	5	8	13	21	34	55	89	144	233

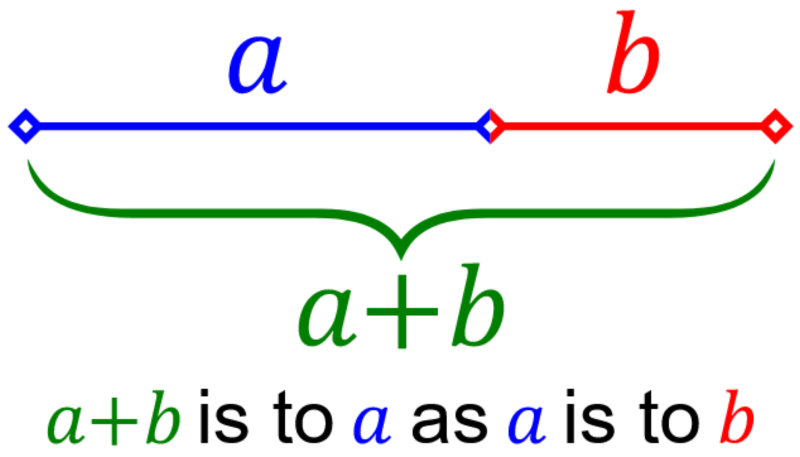
$$F_n = F_{n-1} + F_{n-2}$$

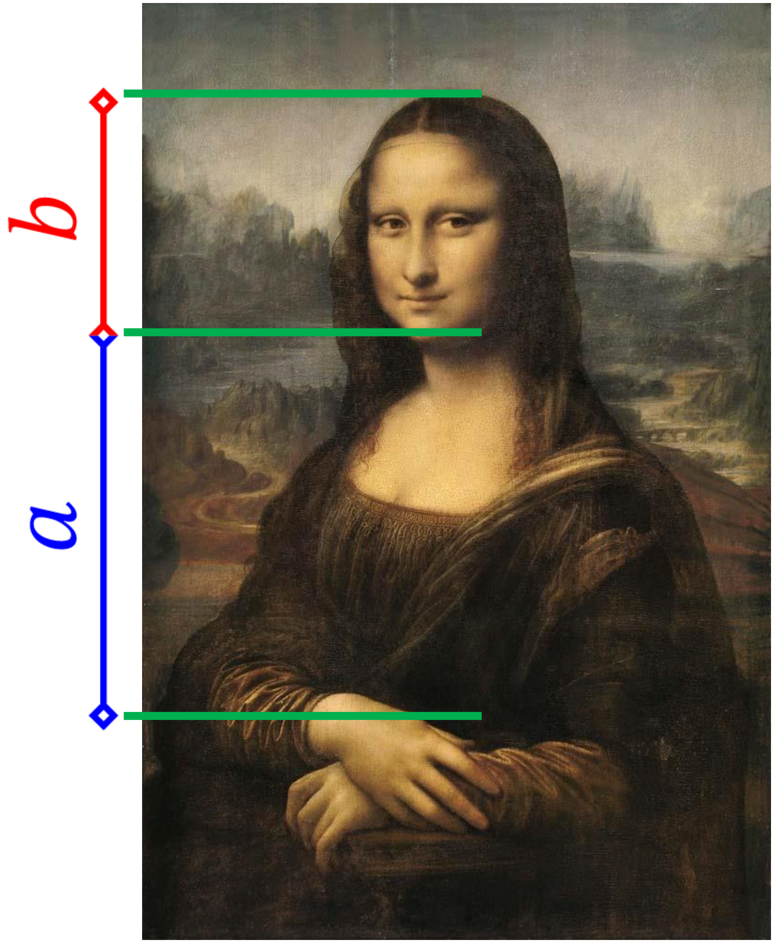
Fibonacci and Golden Ratio



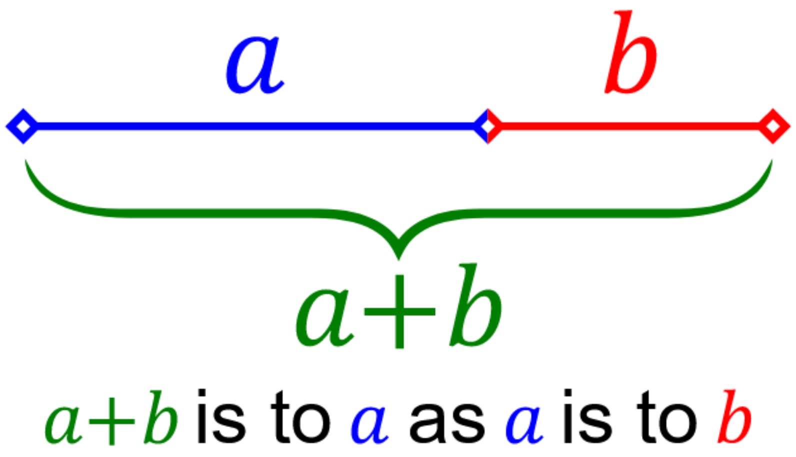
$a+b$ is to a as a is to b

Fibonacci and Golden Ratio



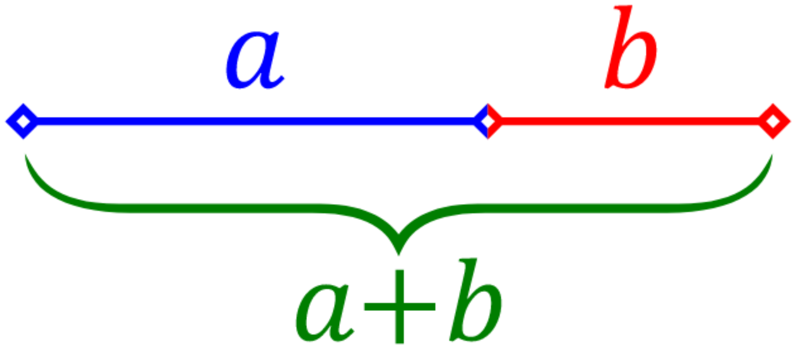


Fibonacci and Golden Ratio



$$\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi$$

Fibonacci and Golden Ratio



$a+b$ is to a as a is to b

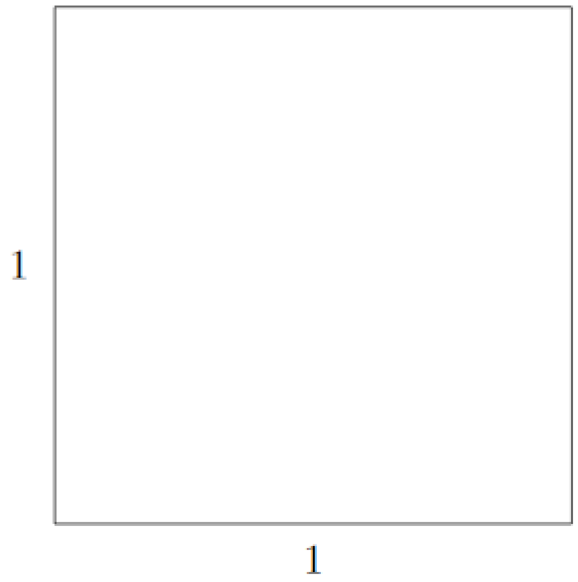
$$\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi$$

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$$

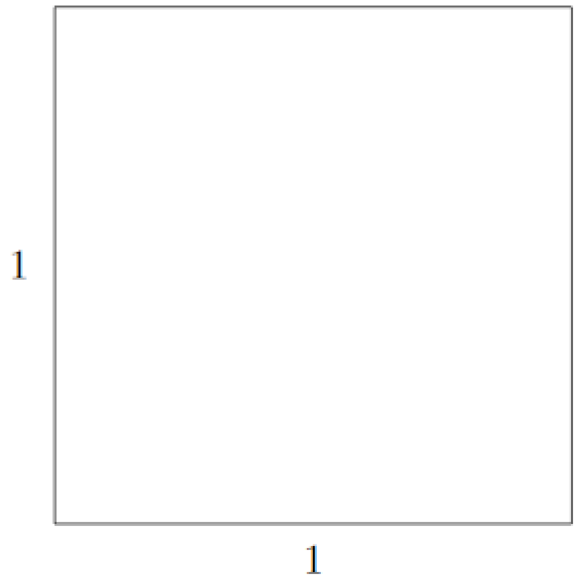
Fibonacci and Golden Ratio

n	F_{n+1}/F_n	value	$F_{n+1}/F_n - \Phi$
1	1/1	1.0000	−0.6180
2	2/1	2.0000	0.3820
3	3/2	1.5000	−0.1180
4	5/3	1.6667	0.0486
5	8/5	1.6000	−0.0180
6	13/8	1.6250	0.0070
7	21/13	1.6154	−0.0026
8	34/21	1.6190	0.0010
9	55/34	1.6176	−0.0004
10	89/55	1.6182	0.0001

Ratio of consecutive Fibonacci numbers approaches Φ.



(a) Construct a square.



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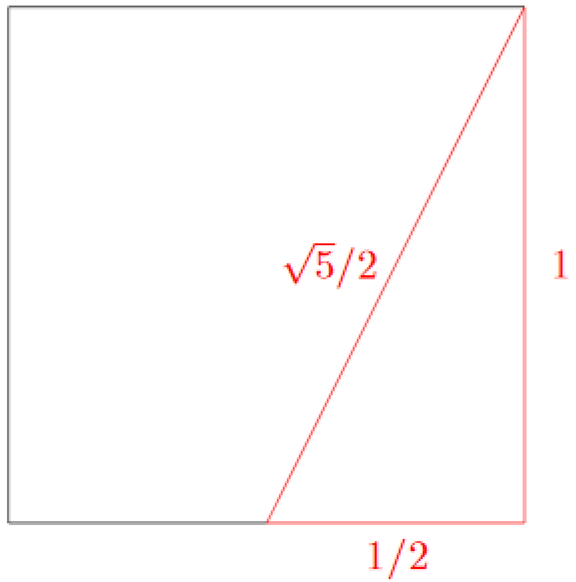
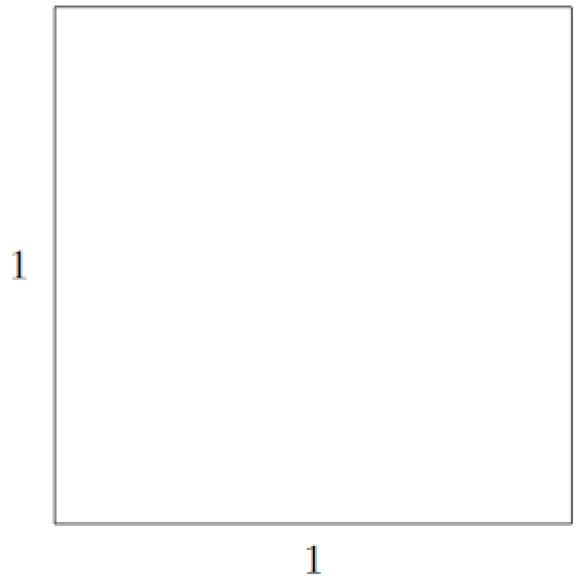


(b) Draw a line from midpoint to corner.



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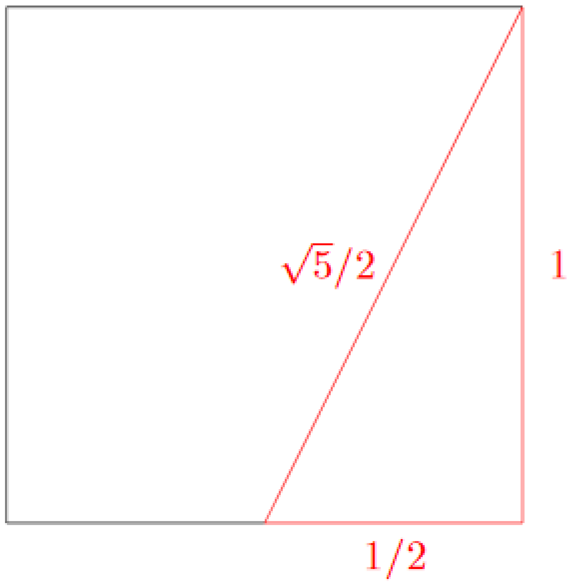
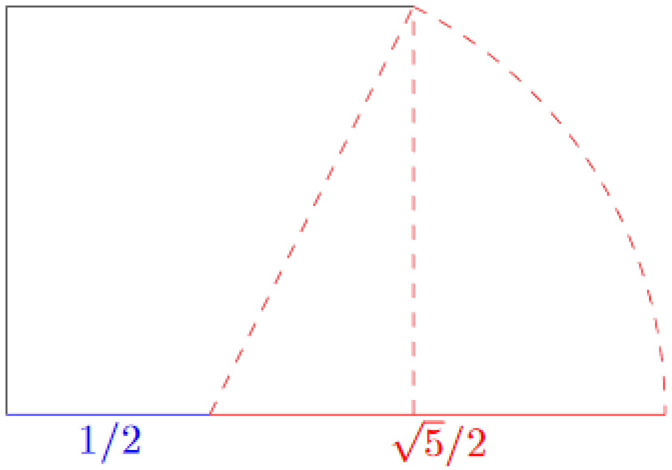
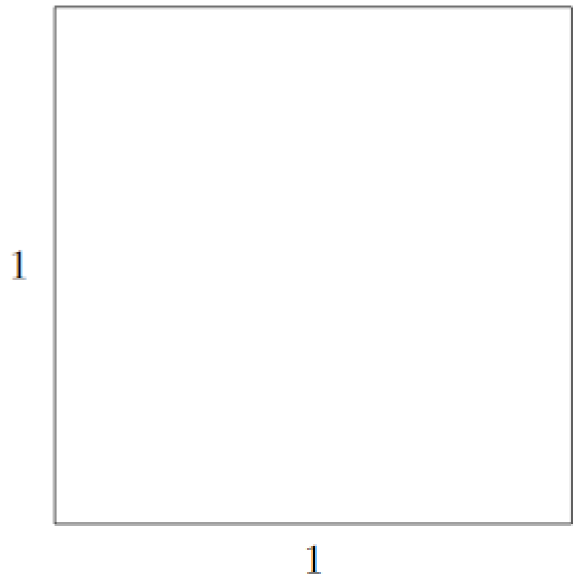


(b) Draw a line from midpoint to corner.



(c) Draw an arc using the internal line as radius.



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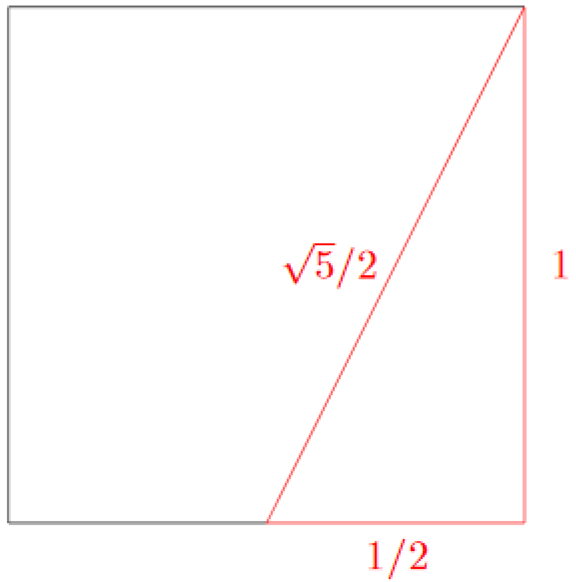
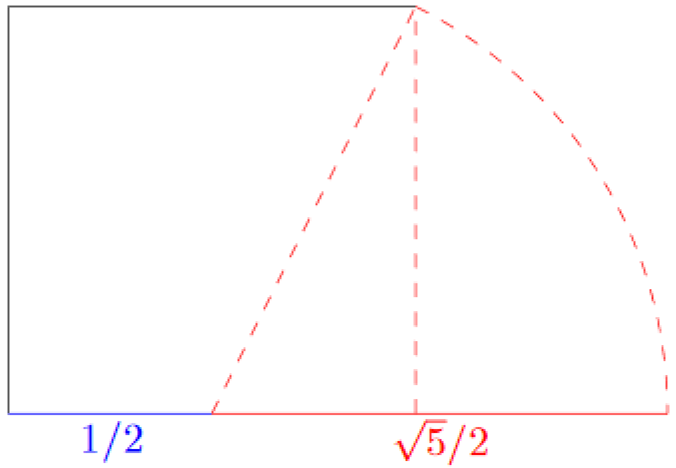


(b) Draw a line from midpoint to corner.



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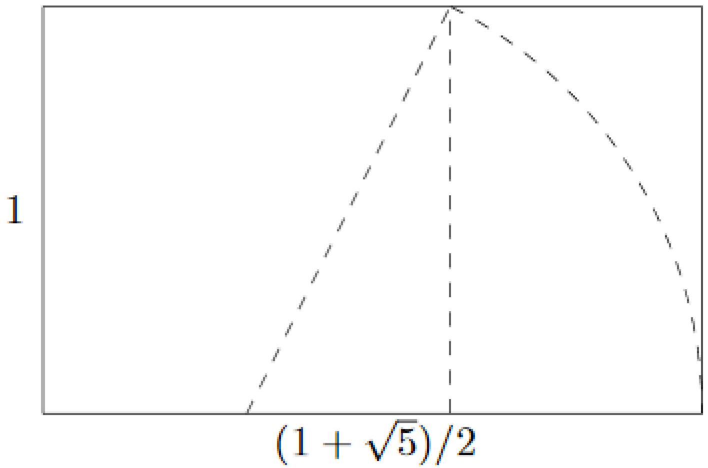
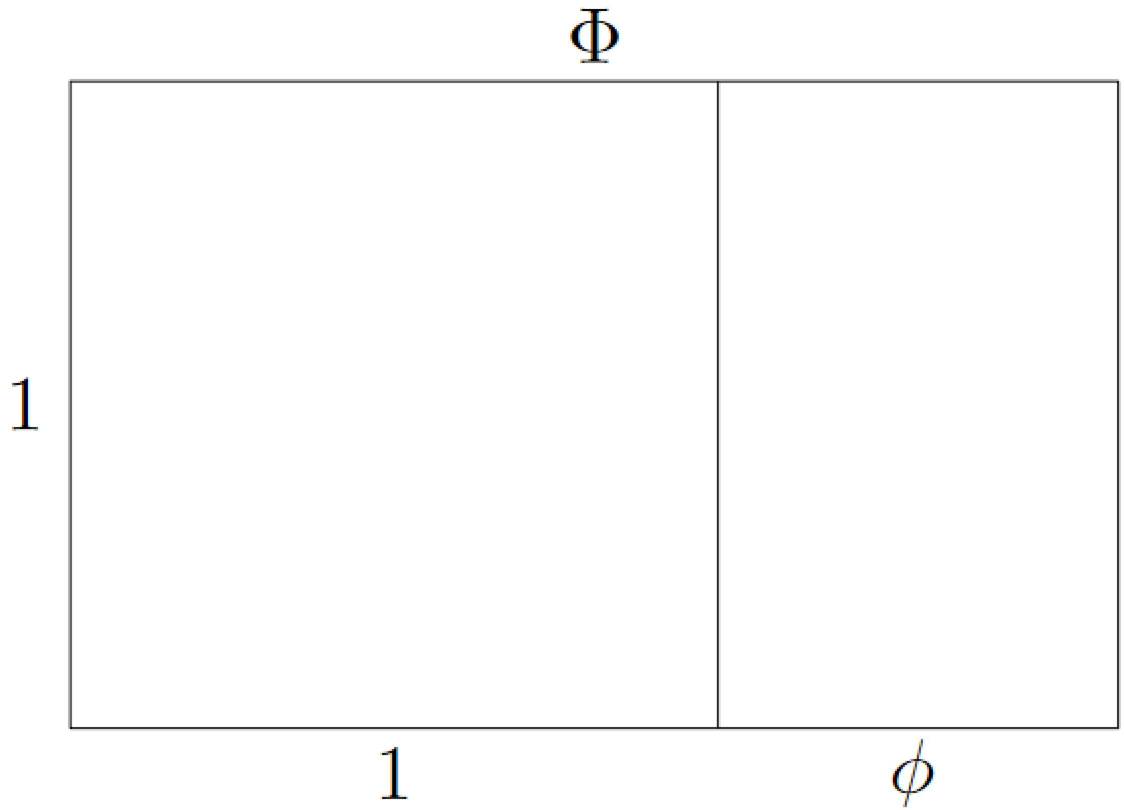
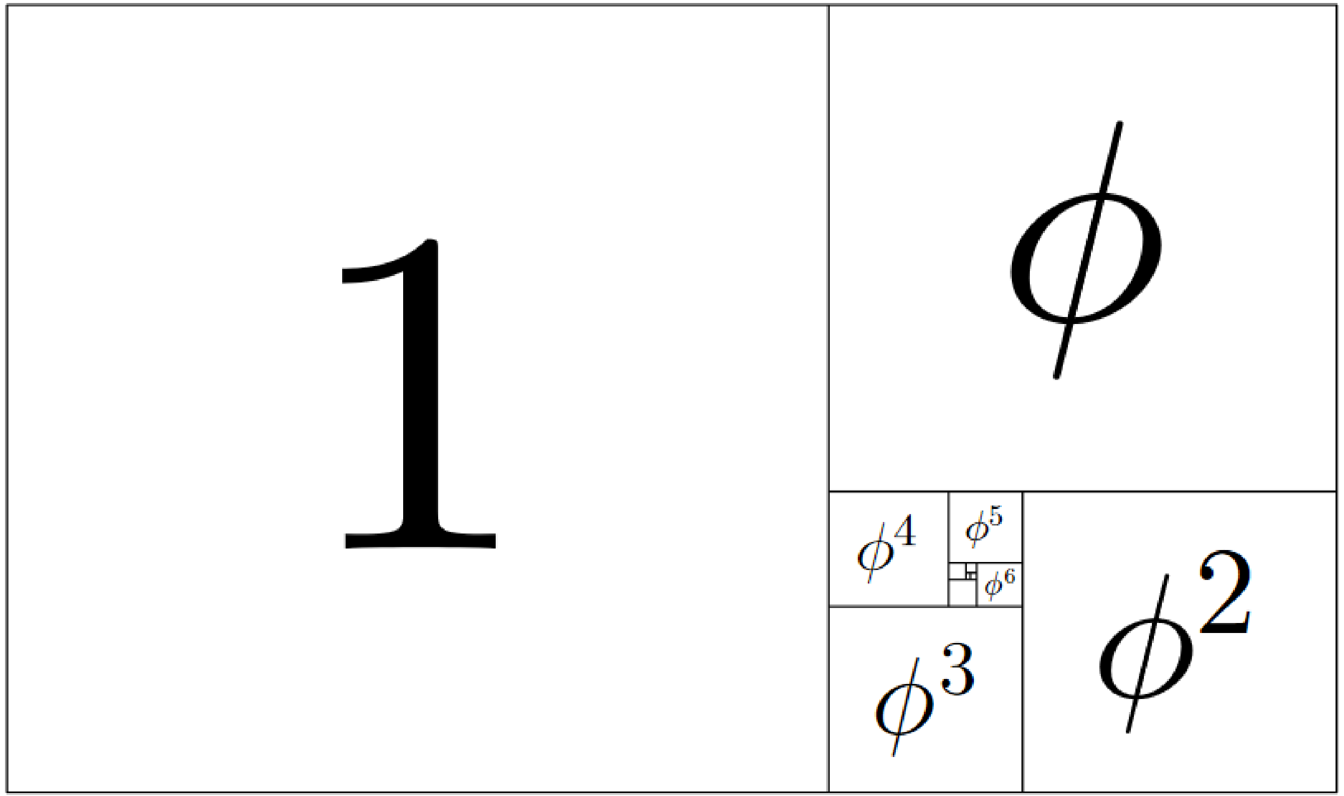


(d) Complete the golden rectangle.

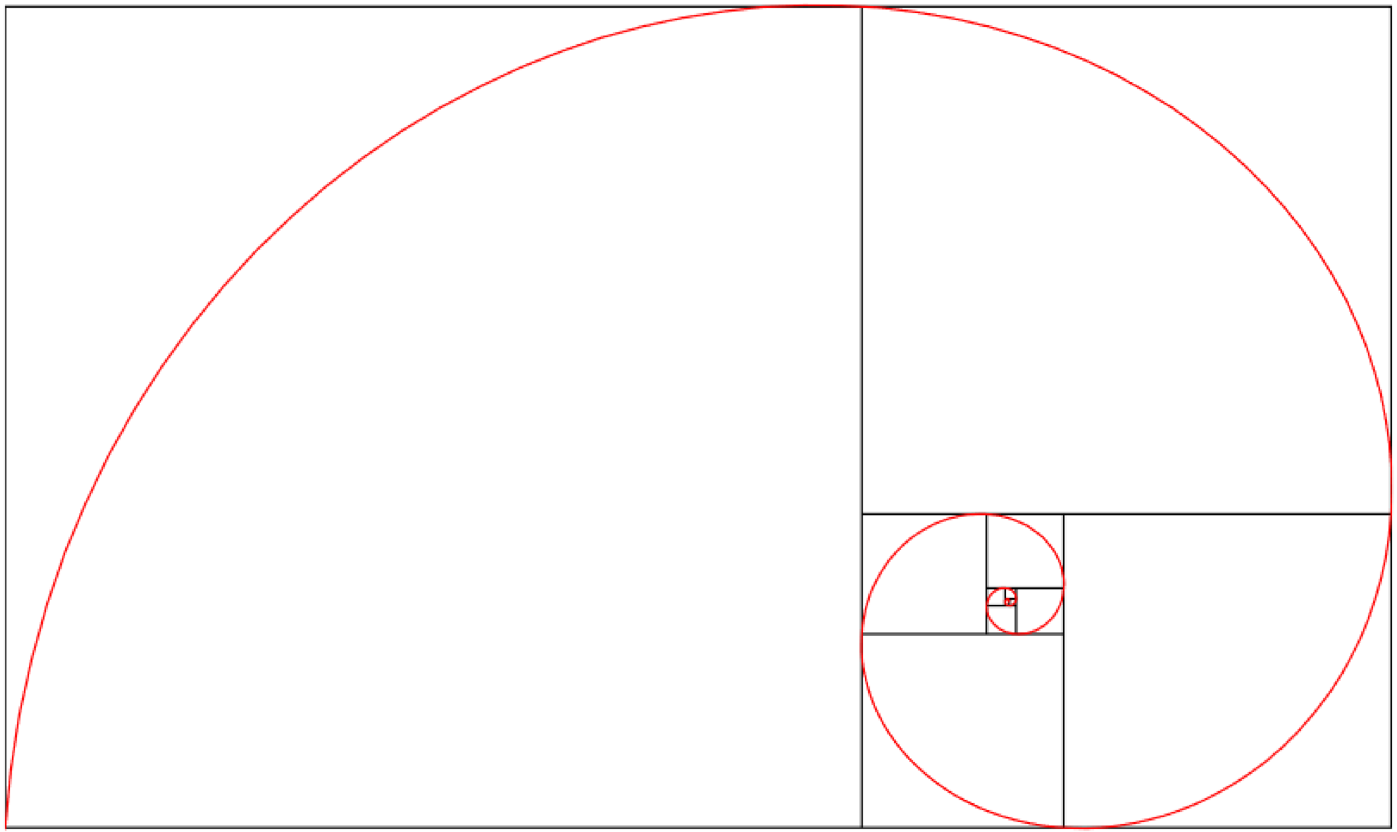


Φ
1
1
ϕ



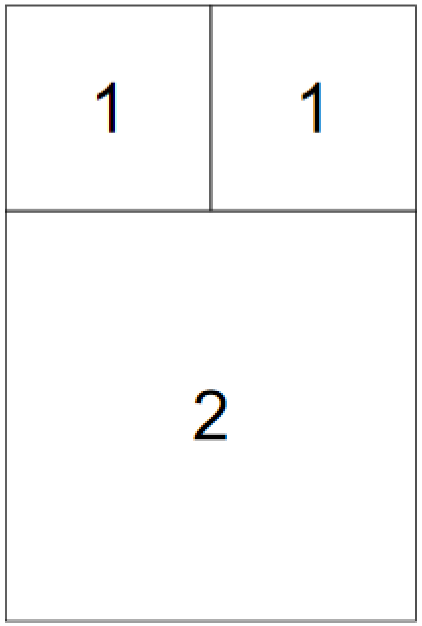
1
ϕ
ϕ^2
ϕ^3
ϕ^4
ϕ^5
ϕ^6

Golden Spiral



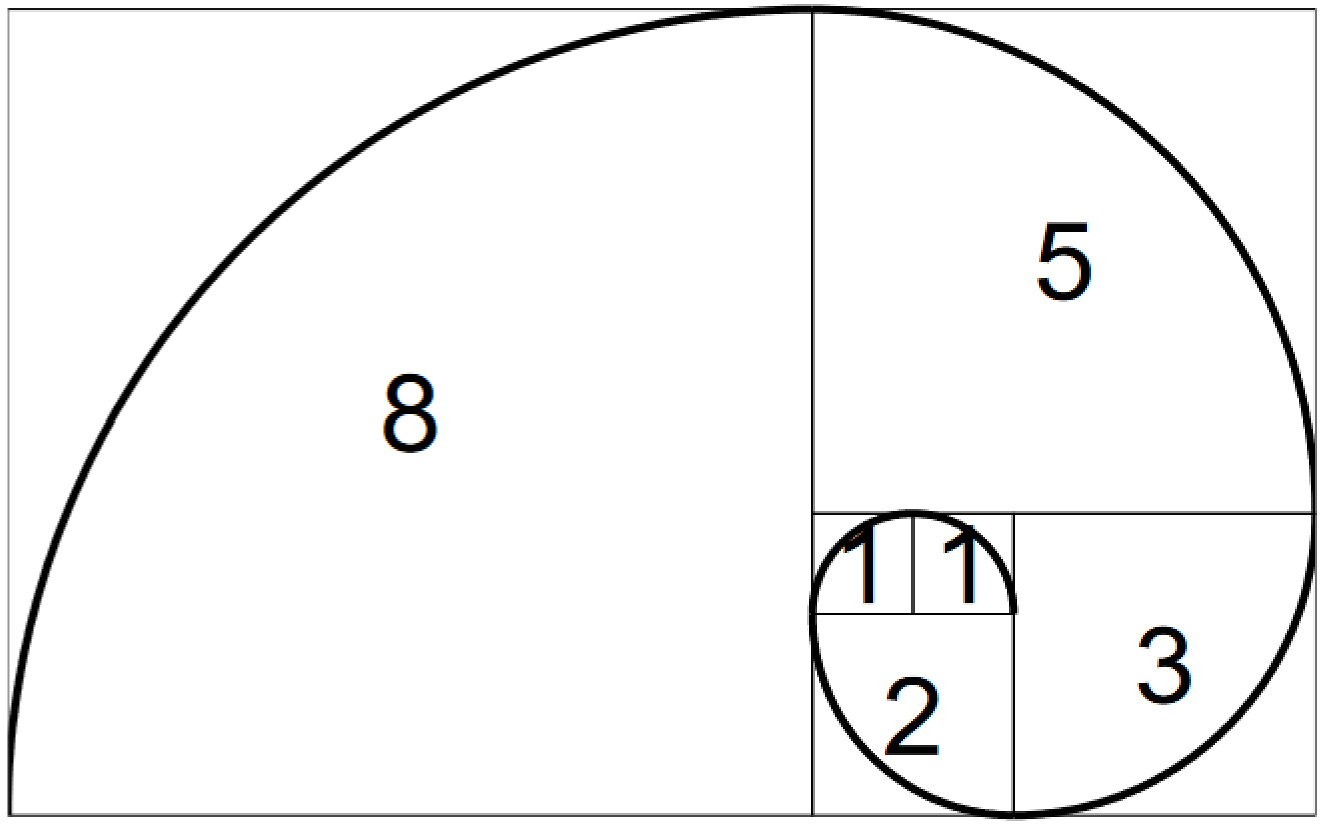
Fibonacci Spiral

(a) $n = 2$: $1^2 + 1^2 = 1 \times 2$.

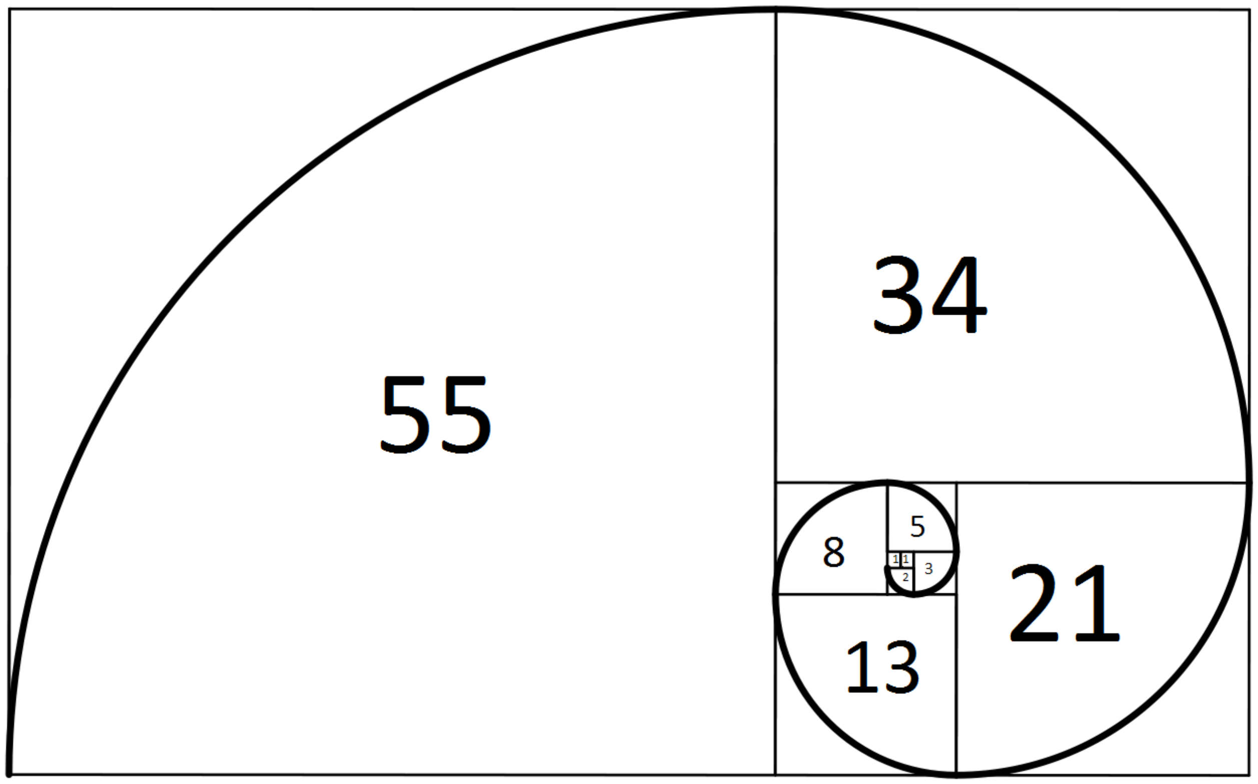


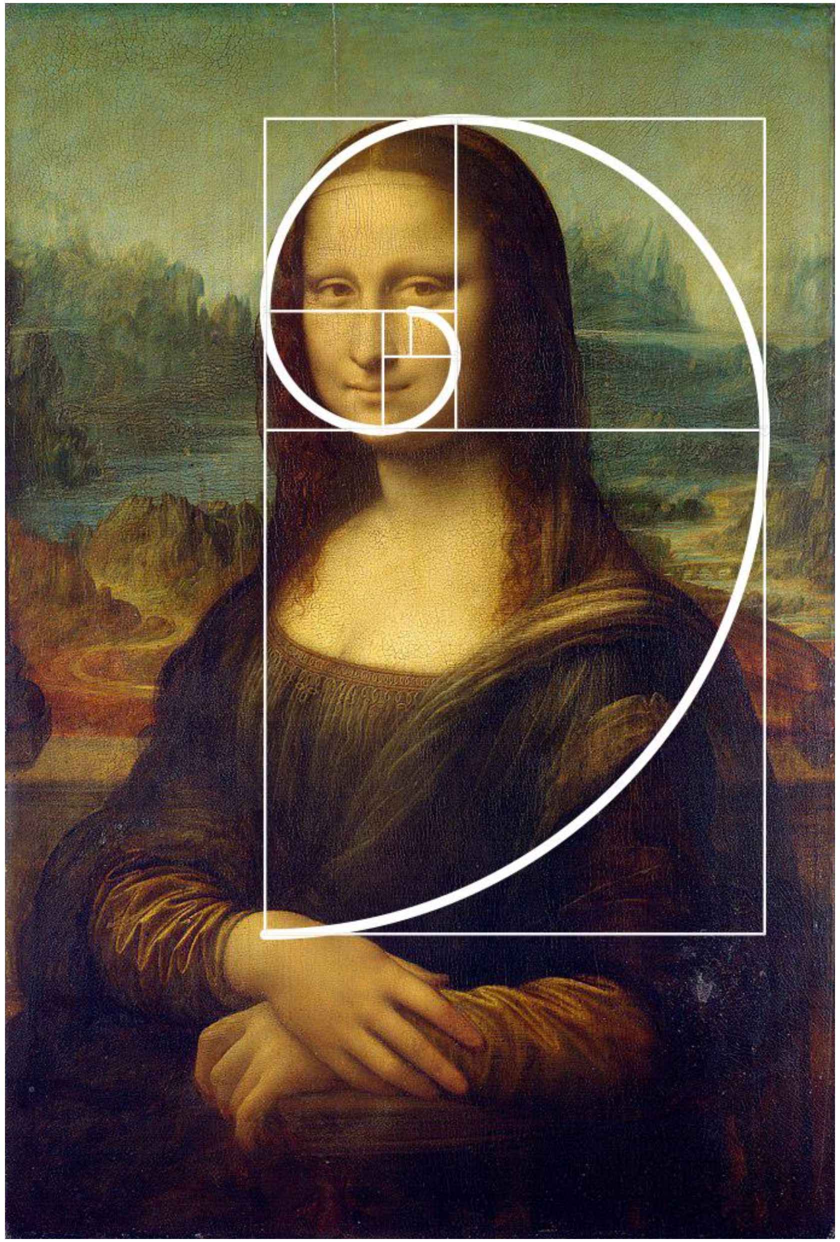
(b) $n = 3$: $1^2 + 1^2 + 2^2 = 2 \times 3$.

Fibonacci Spiral

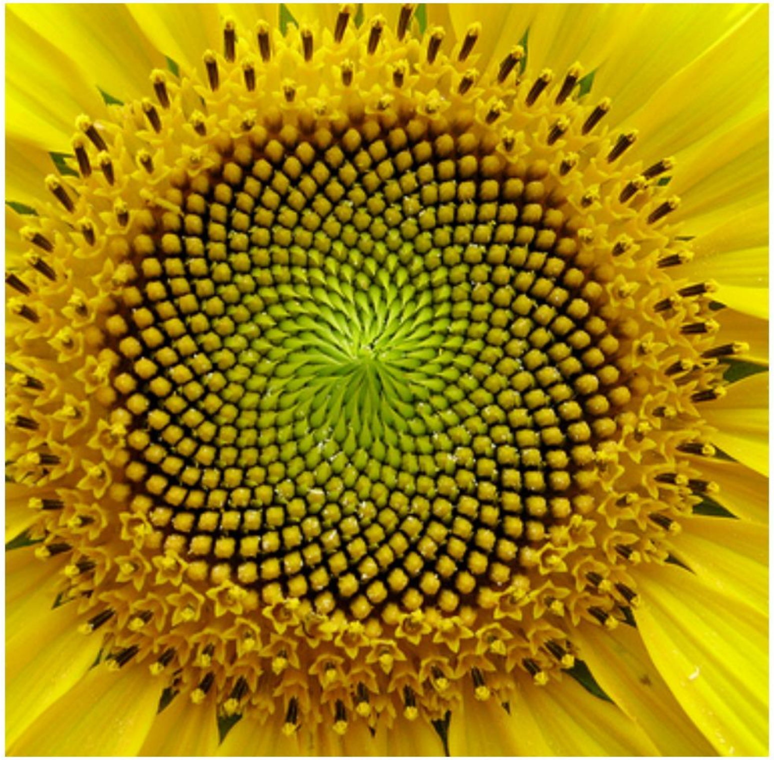


Fibonacci Spiral

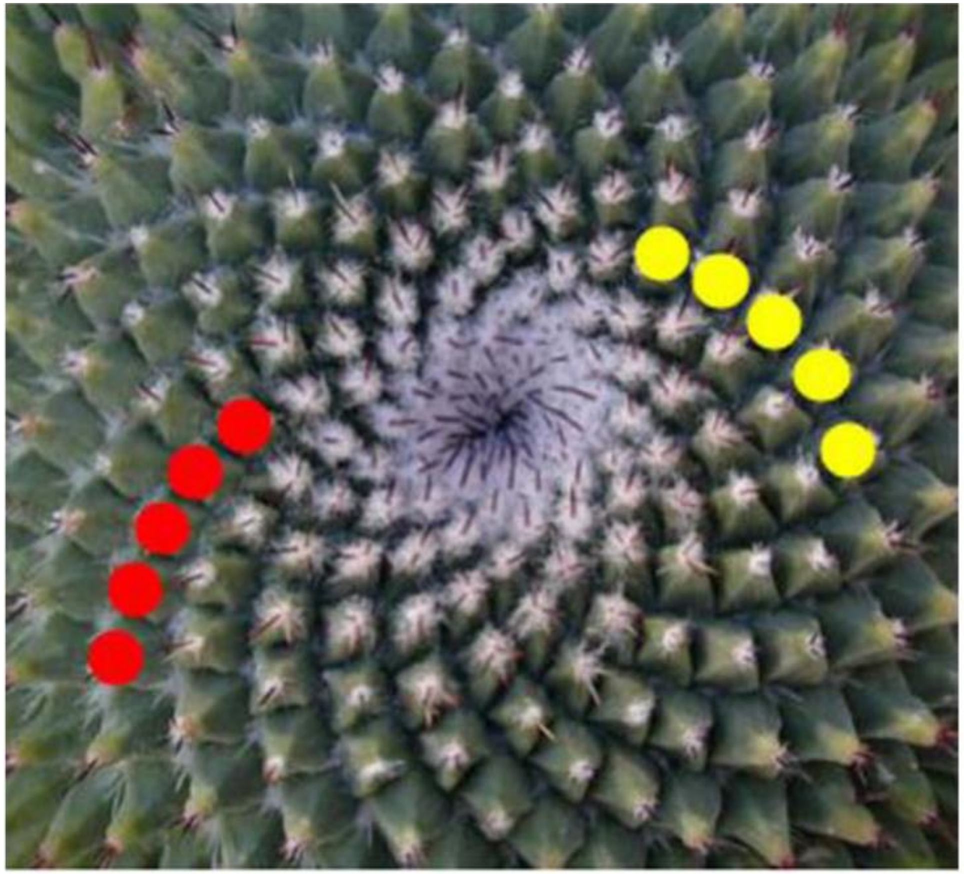




Examples from Nature

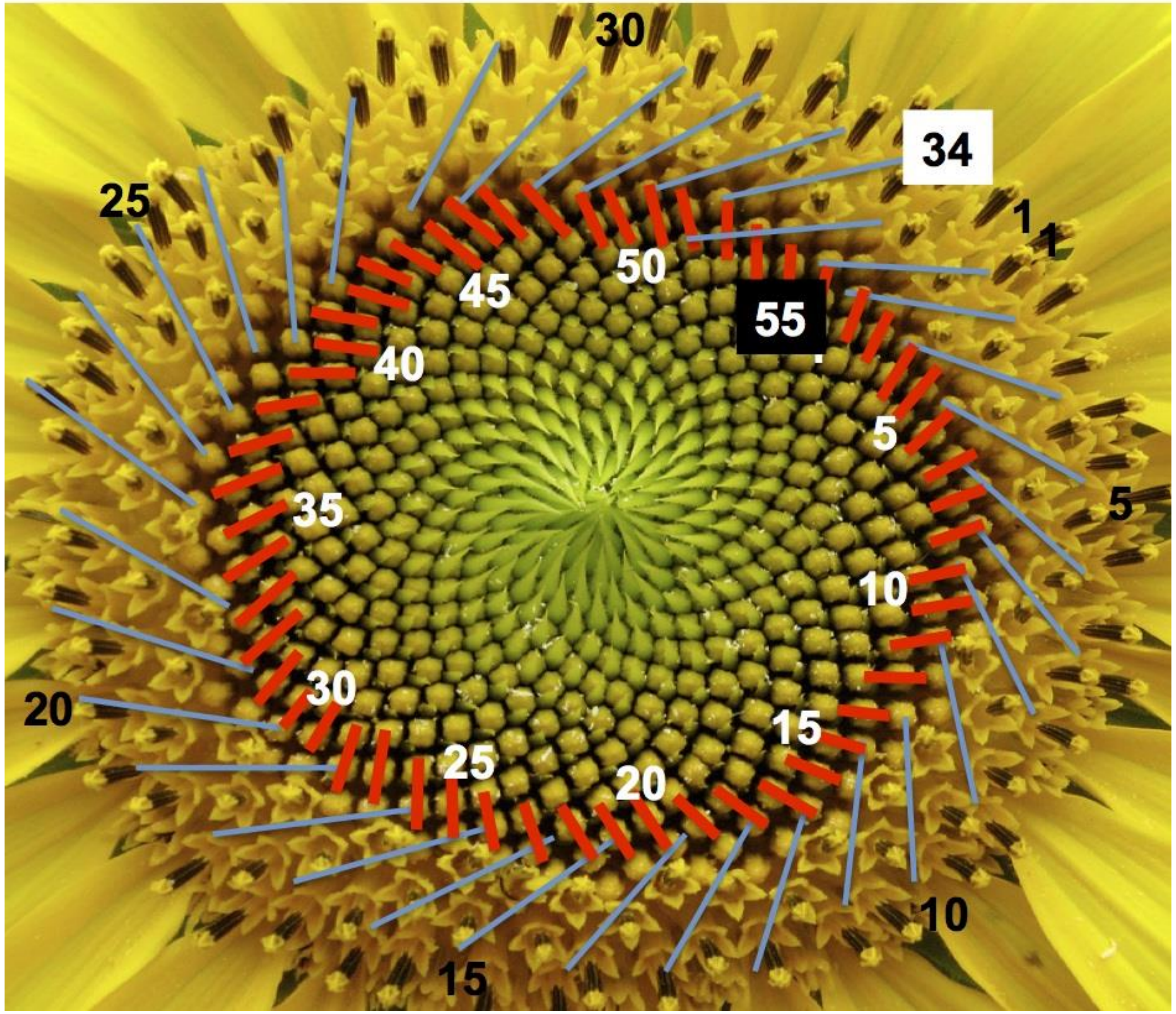




Parastichies



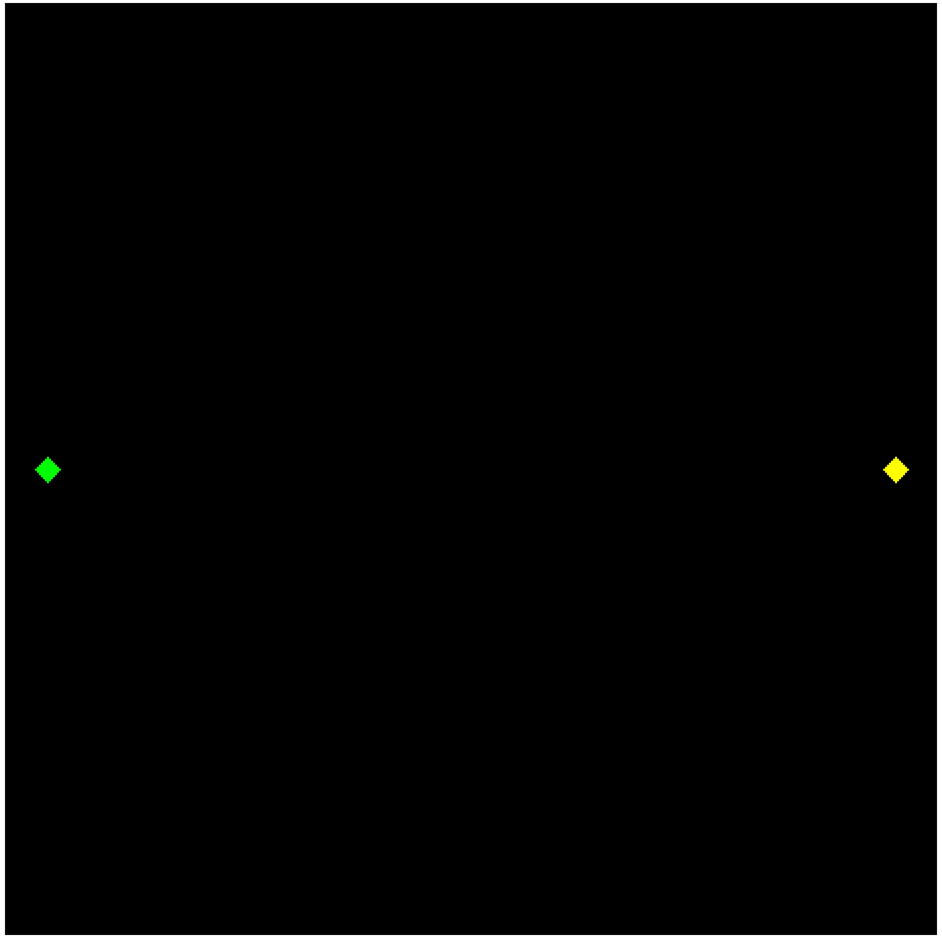
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

Sunflower

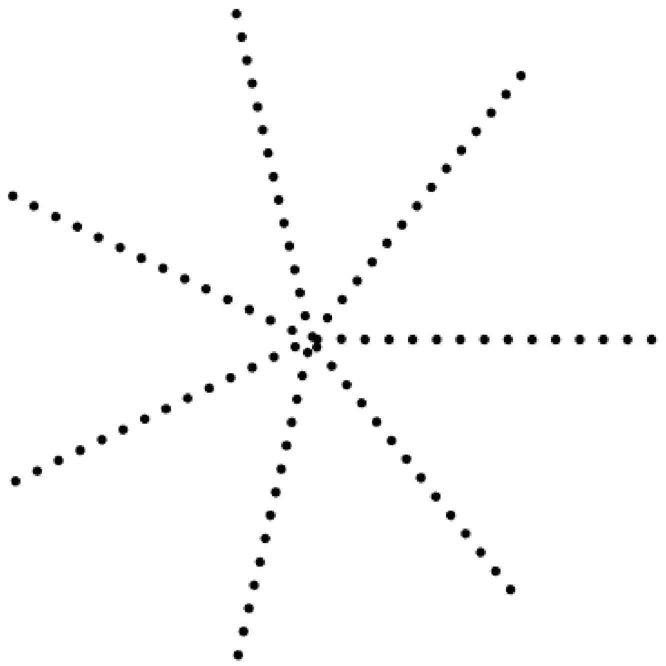


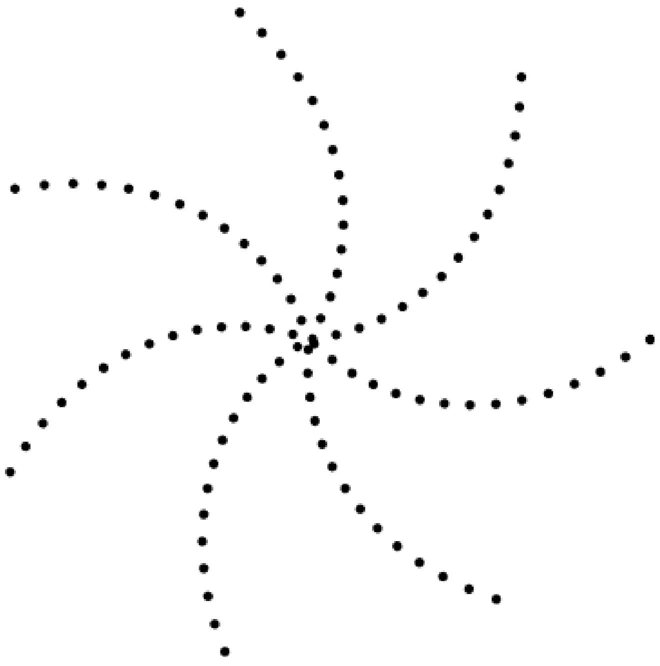
Sunflower Model

- New seeds ae generated in the center
- Seeds move in a linear direction away from the center
- Each new seed selects a new moving direction by rotating with a given angle in respect to the previous seed

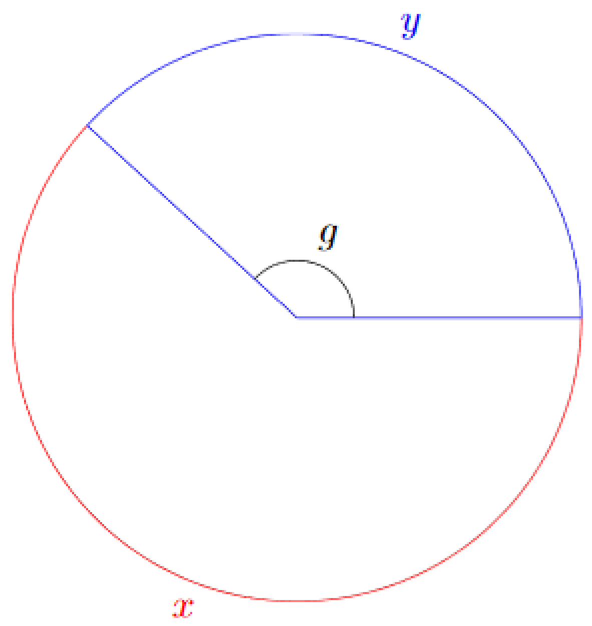


Example Models (1/7) and (PI -3)

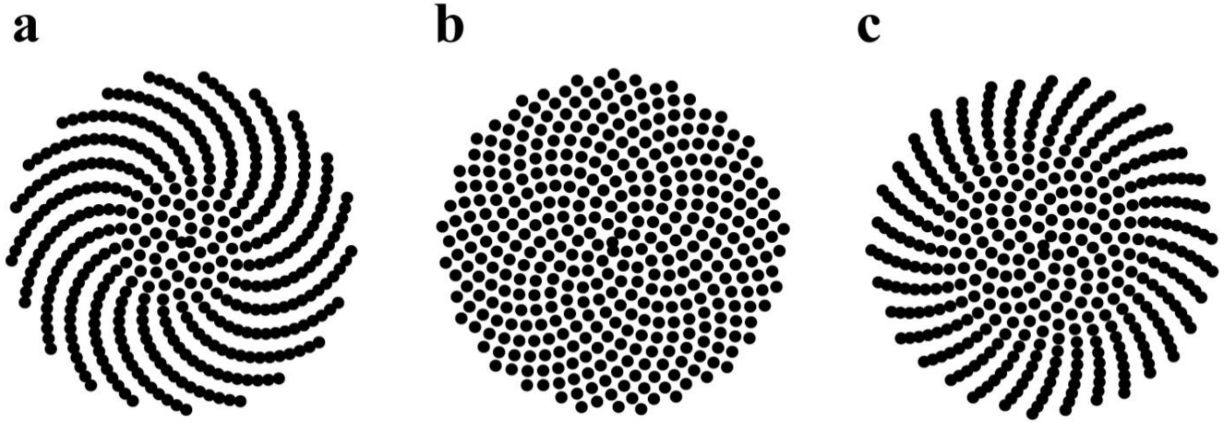




The Golden Angle 137.5



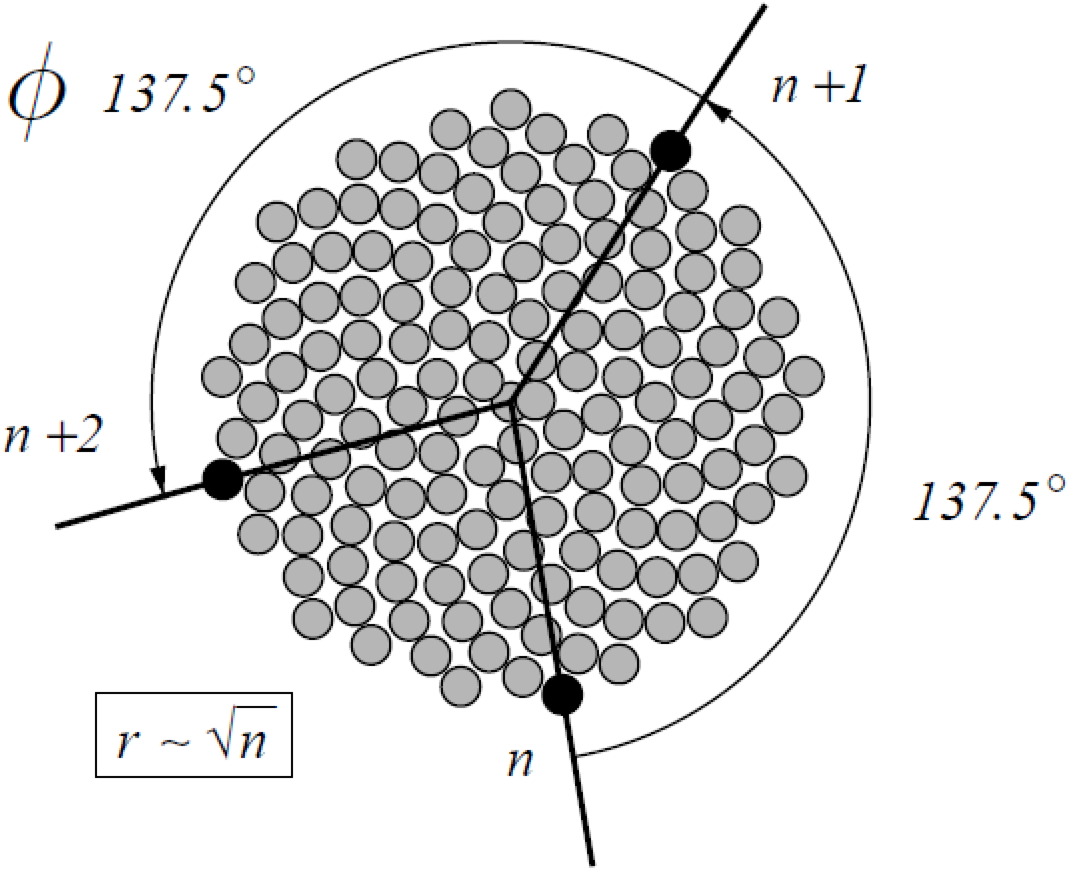
Different divergent angles



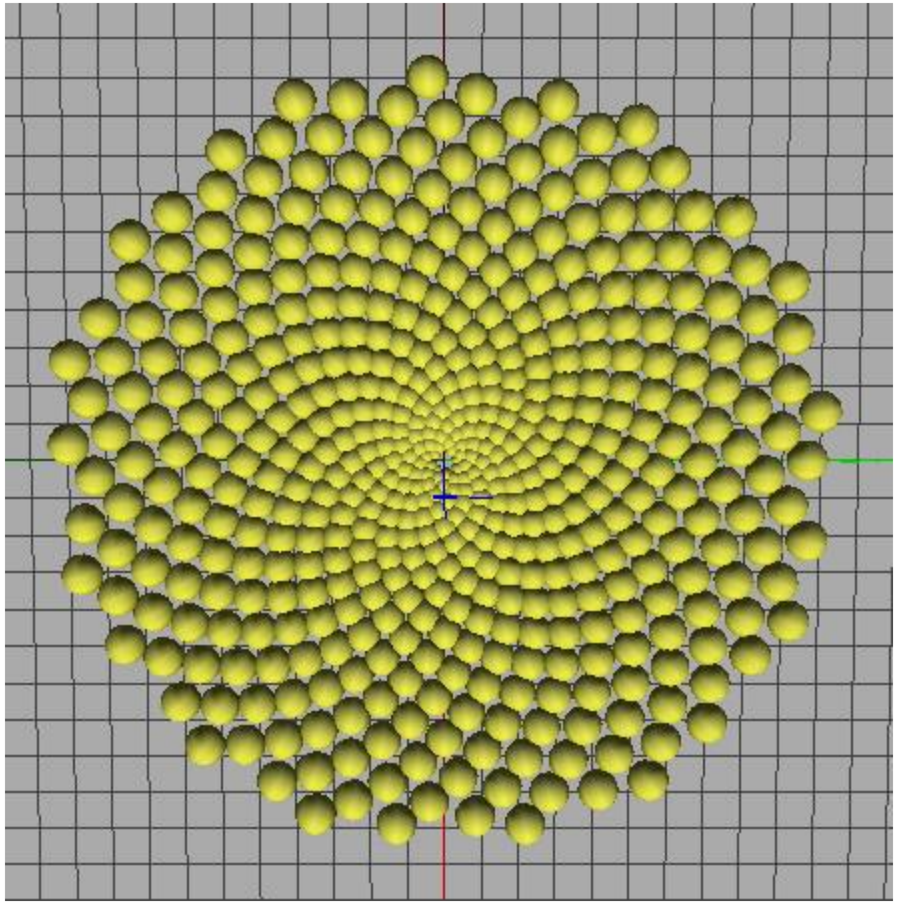
137.3 137.5 137.6

Vogels equation

$$\phi = n * 137.5^\circ, \qquad r = c\sqrt{n}$$

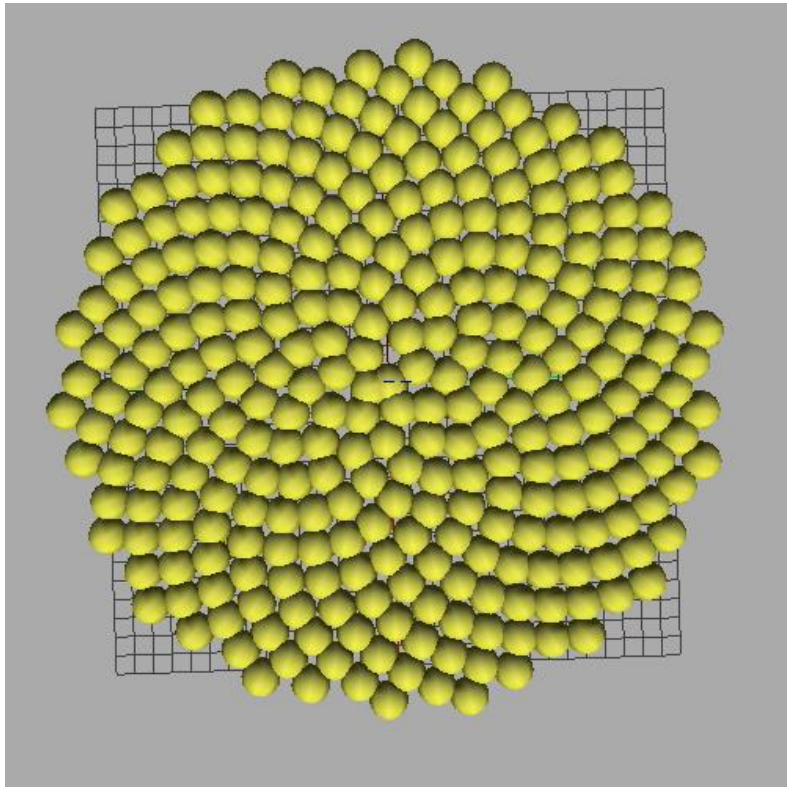


Sunflower Model

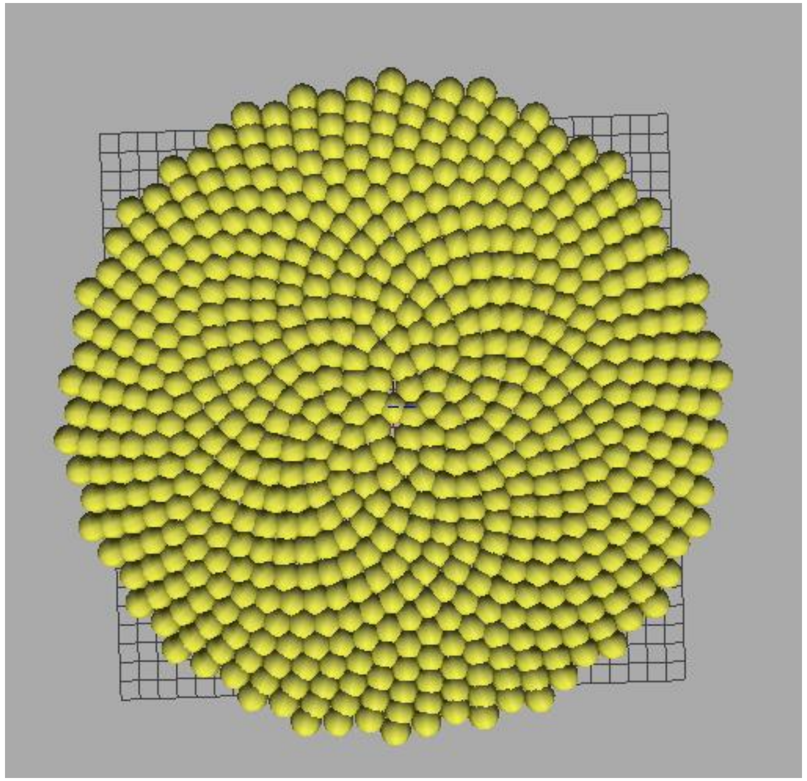


Fibonacci, 137.5

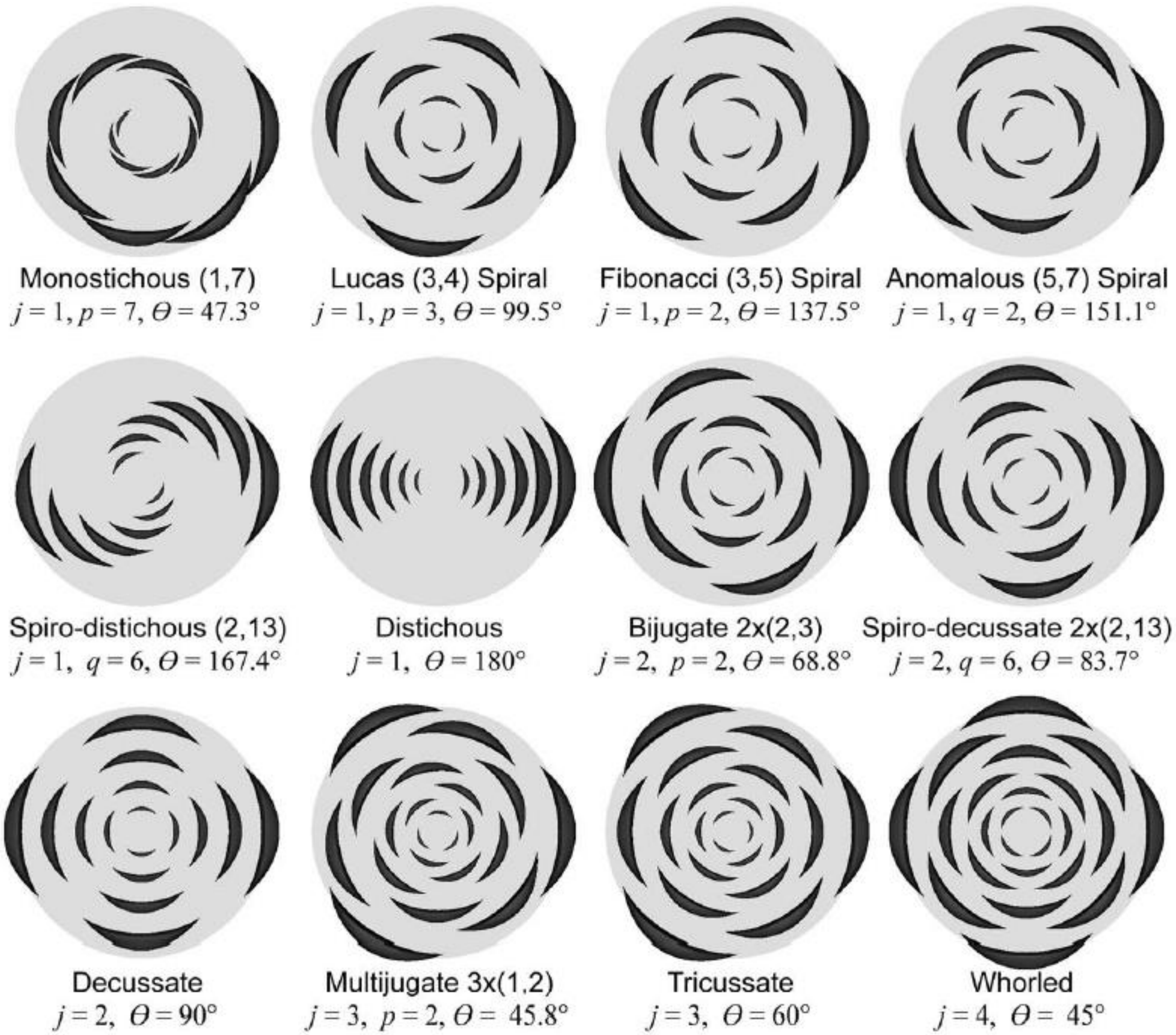
Other known phyllotaxies



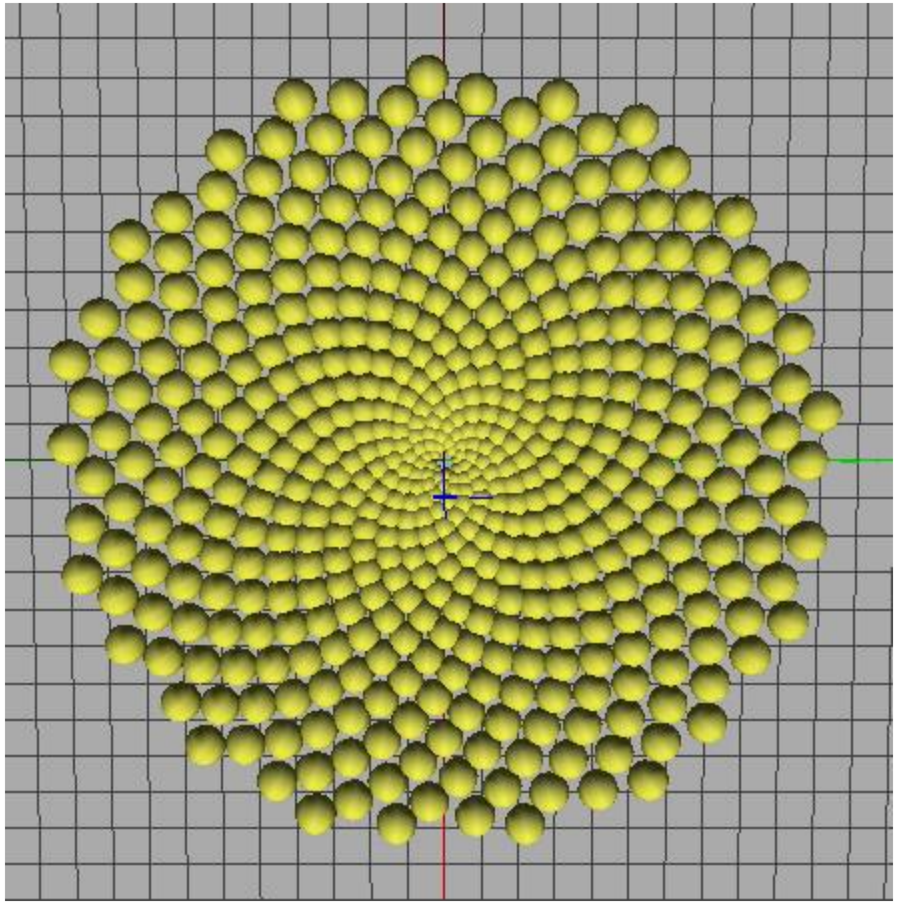
Lucas, 99.5



Bijugate, 68.8


Monostichous (1,7)
$j = 1, p = 7, \Theta = 47.3°$
Lucas (3,4) Spiral
$j = 1, p = 3, \Theta = 99.5°$
Fibonacci (3,5) Spiral
$j = 1, p = 2, \Theta = 137.5°$
Anomalous (5,7) Spiral
$j = 1, q = 2, \Theta = 151.1°$
Spiro-distichous (2,13)
$j = 1, q = 6, \Theta = 167.4°$
Distichous
$j = 1, \Theta = 180°$
Bijugate 2x(2,3)
$j = 2, p = 2, \Theta = 68.8°$
Spiro-decussate 2x(2,13)
$j = 2, q = 6, \Theta = 83.7°$
Decussate
$j = 2, \Theta = 90°$
Multijugate 3x(1,2)
$j = 3, p = 2, \Theta = 45.8°$
Tricussate
$j = 3, \Theta = 60°$
Whorled
$j = 4, \Theta = 45°$

Exercise



Fibonacci, 137.5

Exercise: Ananas Model (cylinder)