PFE 7

Palubicki

Differential equation

- Equations of the form $\frac{dx(t)}{dt} = f\ (x(t), t)$
- when x is a 1D function we call the above an **ordinary differential equation** otherwise a **partial differential equation**

Differential equation

- Equations of the form $\frac{dx(t)}{dt} = \ f\ (x(t), t)$
- when x is a 1D function we call the above an **ordinary differential equation** otherwise a **partial differential equation**
- $\frac{dx(t)}{dt}$ or $\frac{dx}{dt}$ or $x'(t)$ is read as "**the rate of change of x over t**"

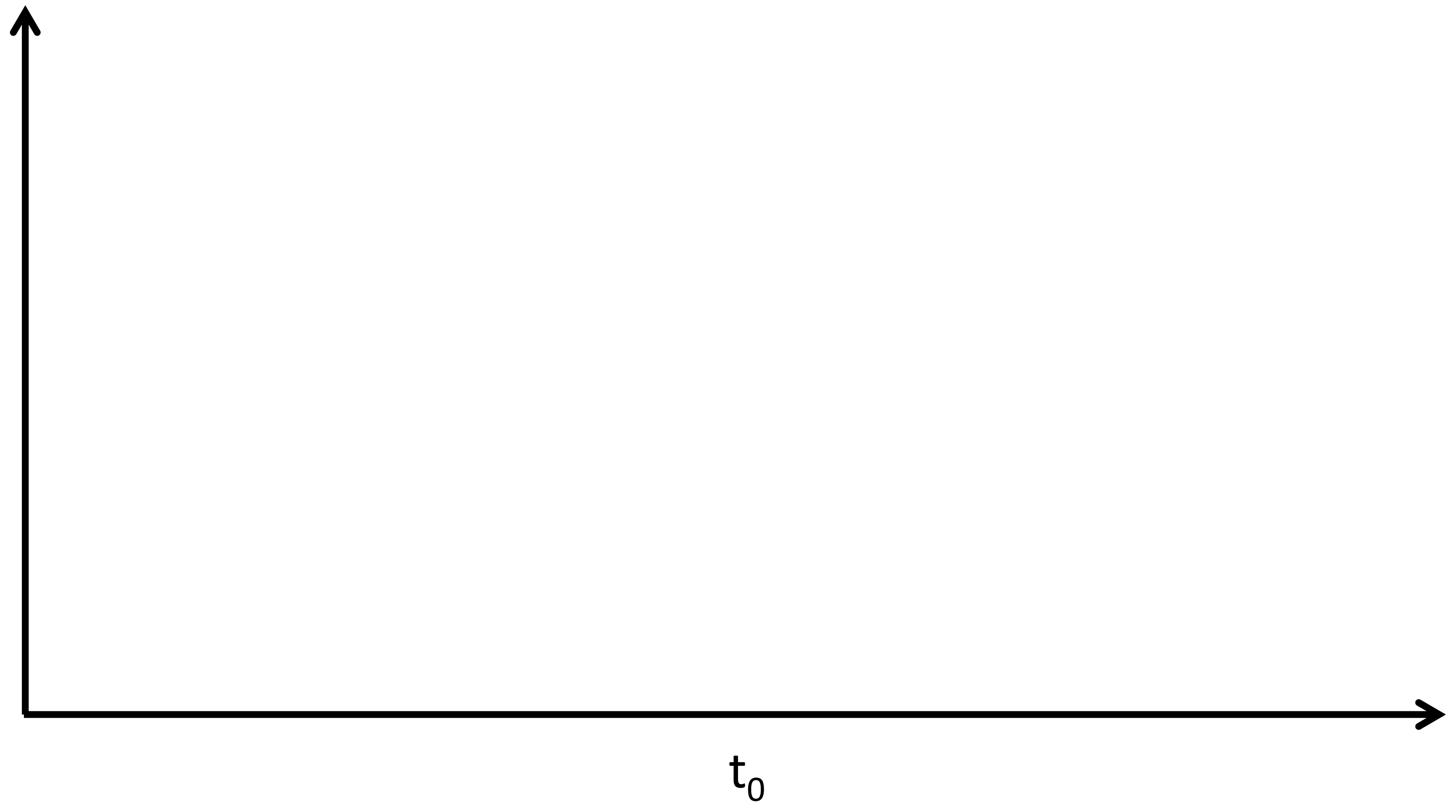
Differential equation

- Equations of the form $\frac{dx(t)}{dt} = f\ (x(t), t)$
- when x is a 1D function we call the above an **ordinary differential equation** otherwise a **partial differential equation**
- $\frac{dx(t)}{dt}$ or $\frac{dx}{dt}$ or $x'(t)$ is read as "**the rate of change of x over t**"
- f is the function that computes the derivative of x with respect to t

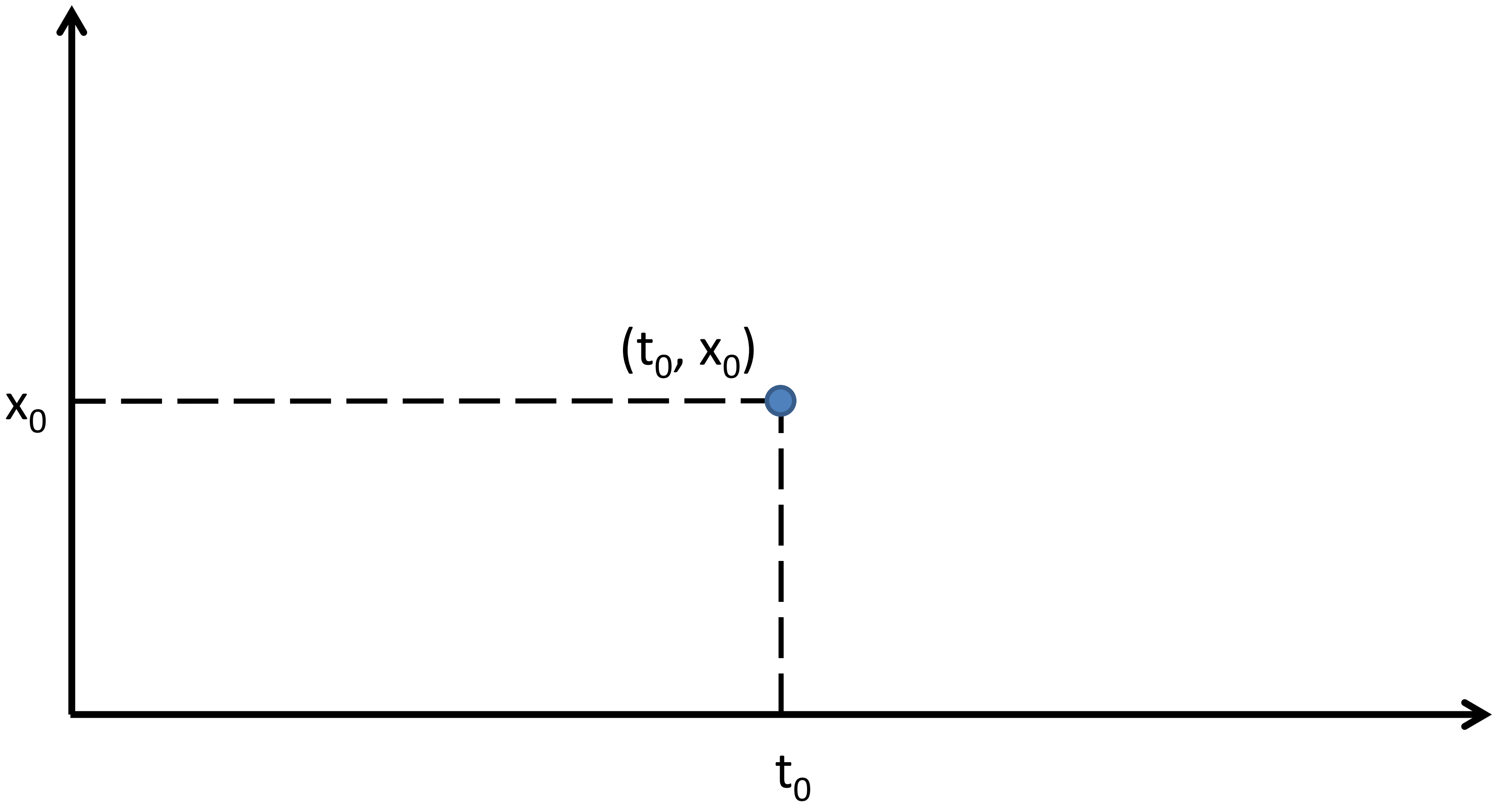
Differential equation

- Equations of the form $\frac{dx(t)}{dt} = \ f\ (x(t), t)$
- when x is a 1D function we call the above an **ordinary differential equation** otherwise a **partial differential equation**
- $\frac{dx(t)}{dt}$ or $\frac{dx}{dt}$ or $x'(t)$ is read as "**the rate of change of x over t**"
- f is the function that computes the derivative of x with respect to t
- The solution to a differential equation is a **function x** that satisfies the equation

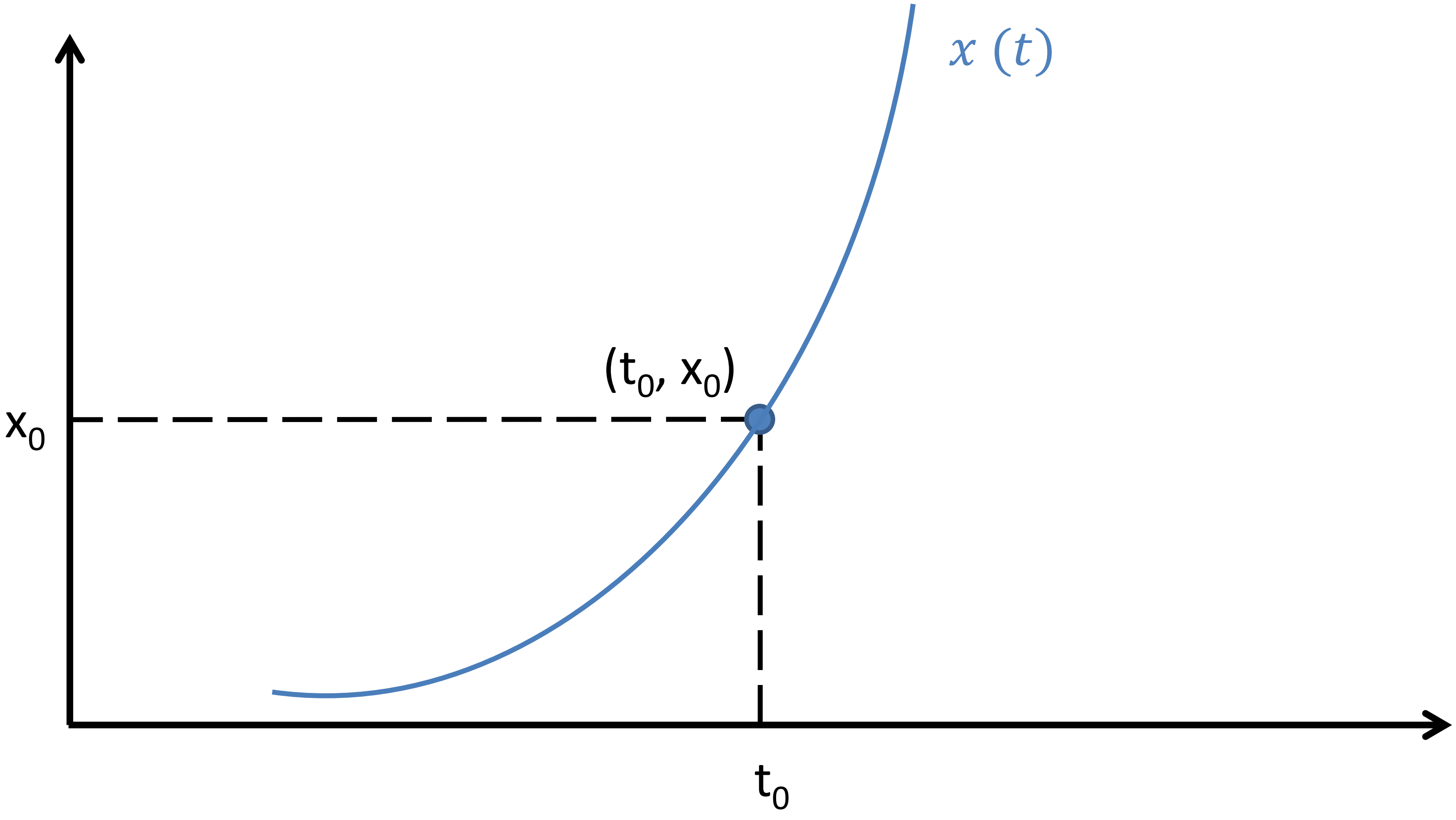
Geometric Interpretation (1st order ODE)



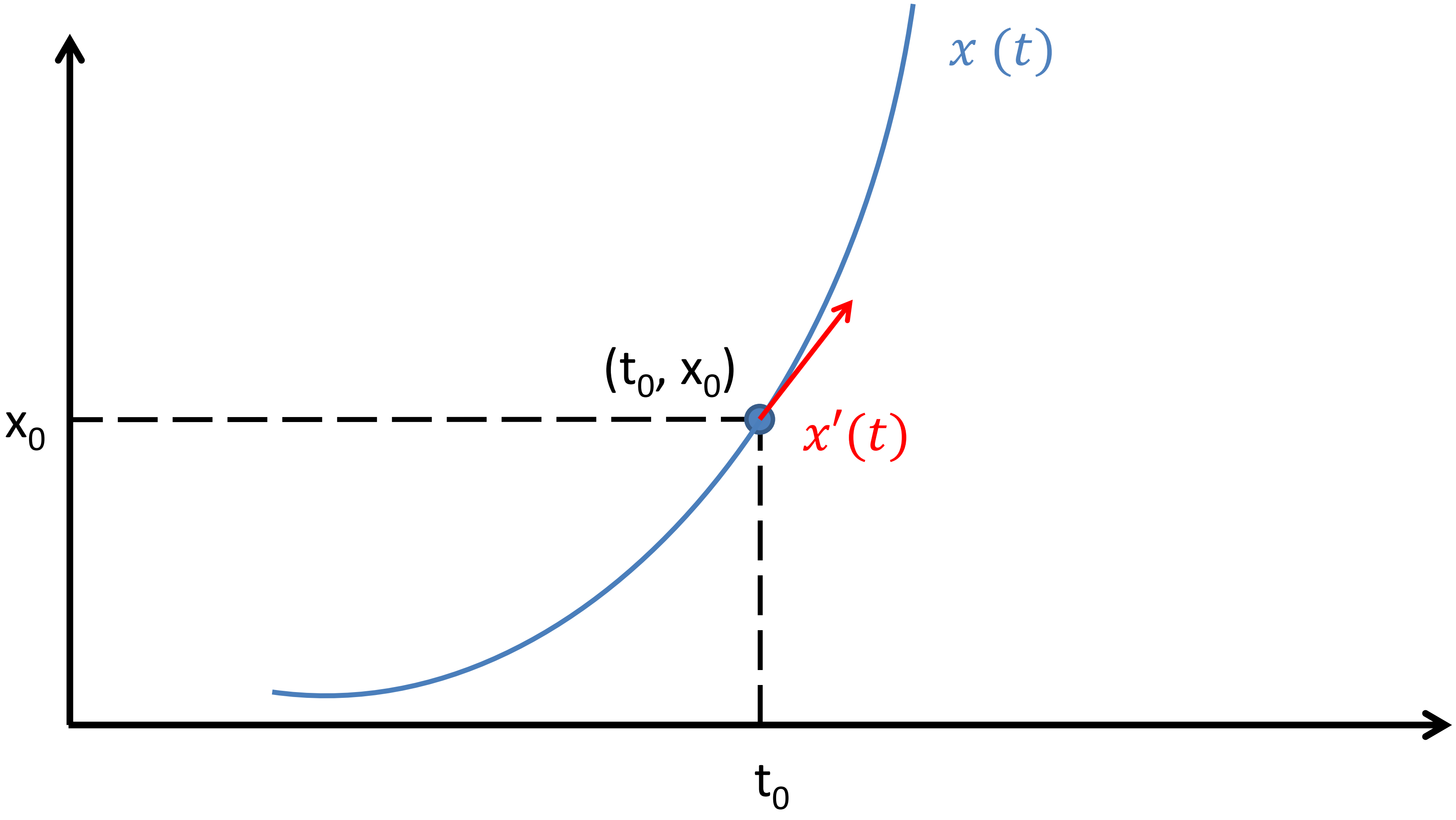
Geometric Interpretation (1st order ODE)



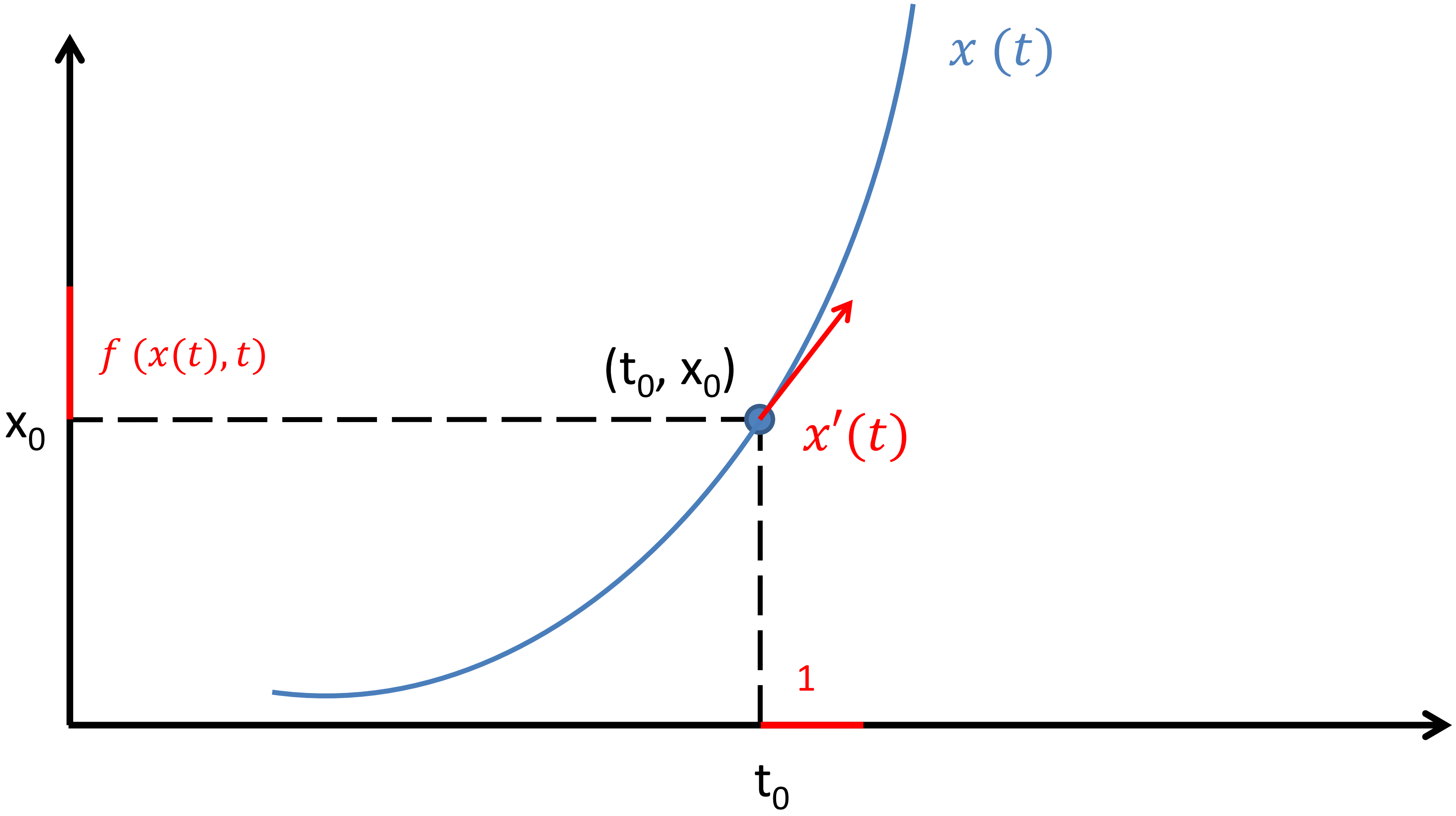
Geometric Interpretation (1^{st} order ODE)



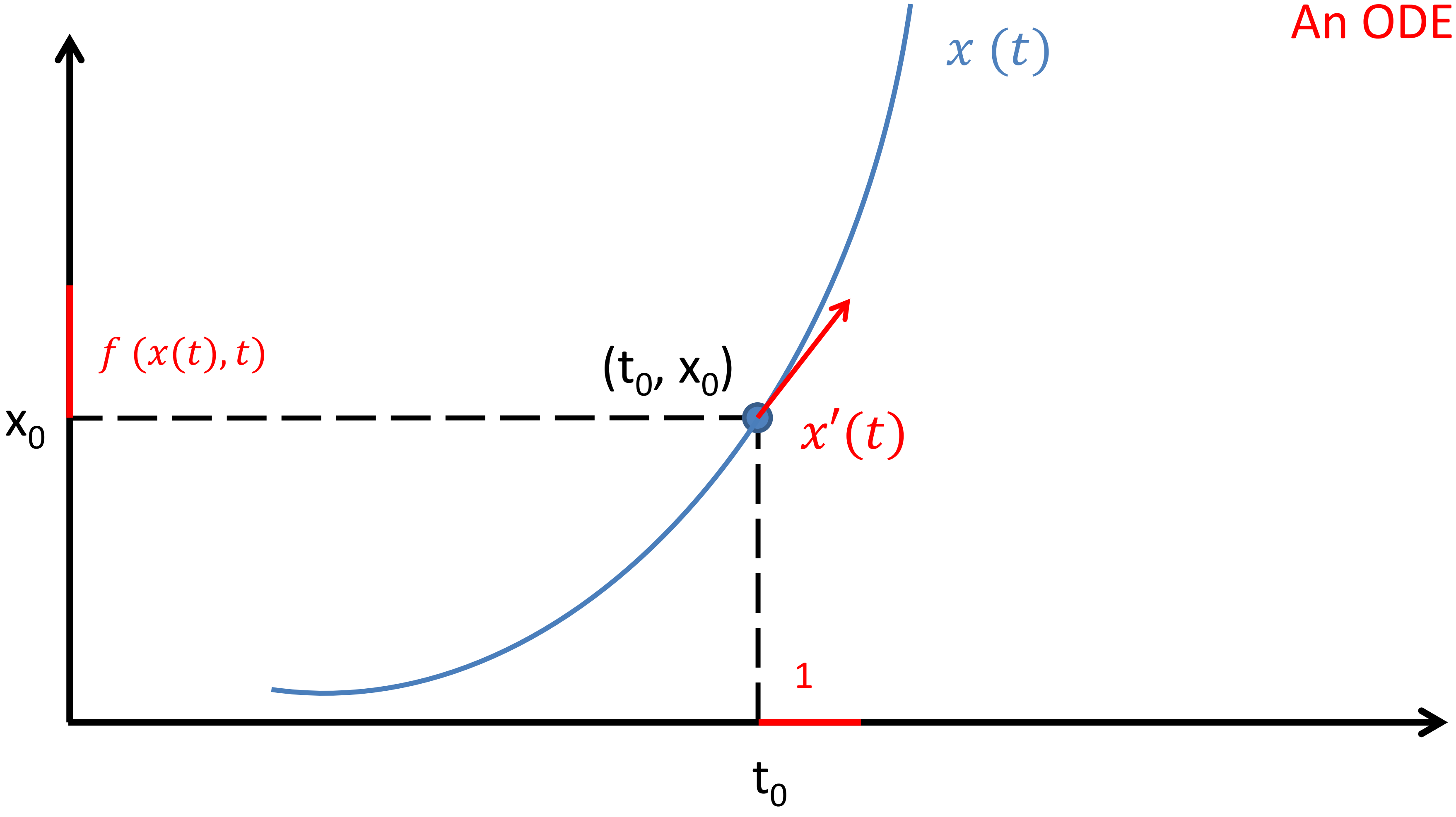
Geometric Interpretation (1^{st} order ODE)



Geometric Interpretation (1st order ODE)

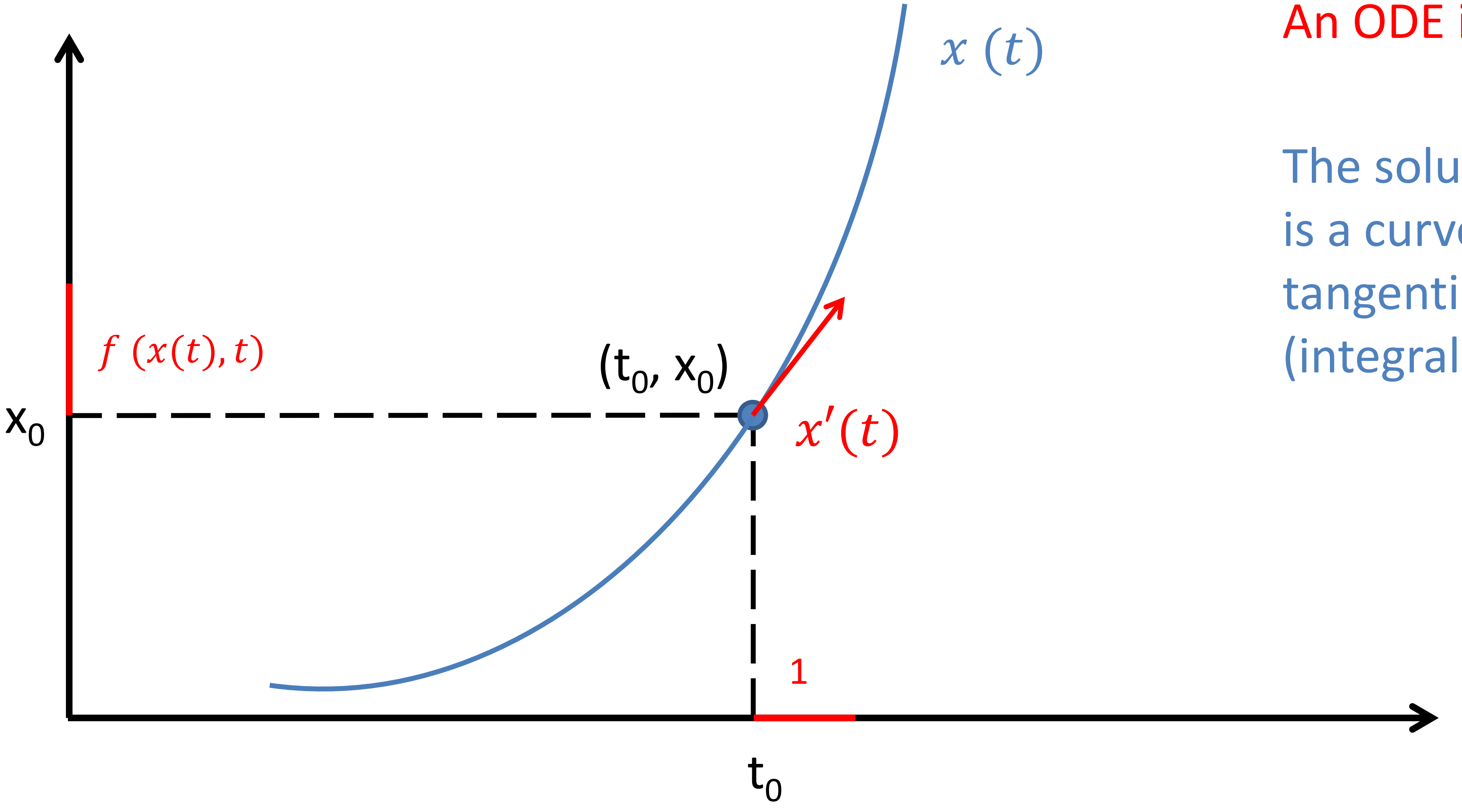


Geometric Interpretation (1^{st} order ODE)



An ODE is a vector field

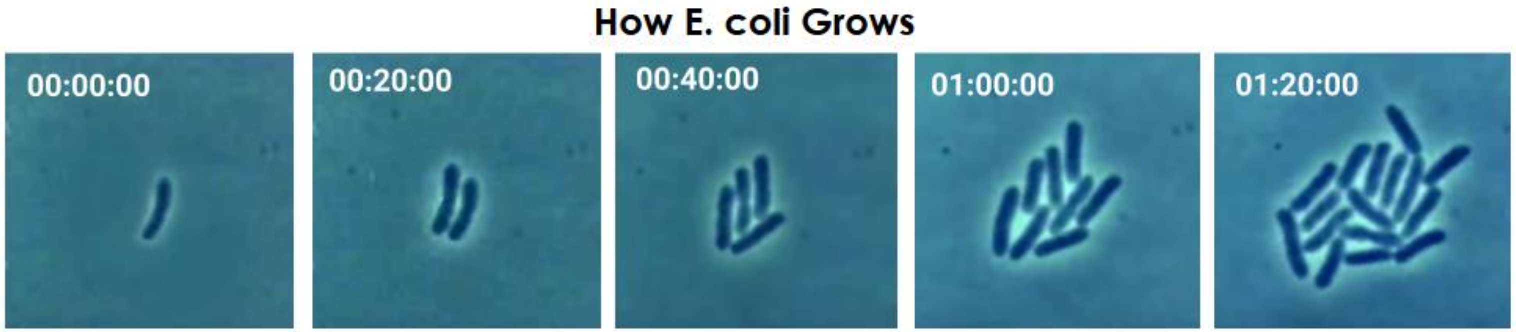
Geometric Interpretation (1st order ODE)



An ODE is a vector field

The solution to an ODE is a curve which is tangential at all points, (integral curve)

Bacterial Growth



1 2 4 8 16

Modeling with differential equations

- $\mathrm{P} = \mathrm{population}, \mathrm{t} = \mathrm{time(days)}$

Modeling with differential equations

- $\mathrm{P} = \mathrm{population}, \mathrm{t} = \mathrm{time(days)}$
- $\frac{dP}{dt} =$

Modeling with differential equations

- $\mathrm{P} = \text{population}, \mathrm{t} = \text{time(days)}$
- $\frac{dP}{dt} = KP$

Modeling with differential equations

- $\mathrm{P} = \mathrm{population}, \mathrm{t} = \mathrm{time(days)}$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$

Modeling with differential equations

- $\mathrm{P} = \text{population}, \mathrm{t} = \text{time(days)}$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$

Modeling with differential equations

- $\mathrm{P} = \mathrm{population}, \mathrm{t} = \mathrm{time(days)}$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$

Modeling with differential equations

- $\mathrm{P} = \mathrm{population}, \mathrm{t} = \mathrm{time(days)}$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$

Modeling with differential equations

- $\mathrm{P} = \text{population}, \mathrm{t} = \text{time(days)}$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$
- $|P| = e^{Kt+C_1}$

Modeling with differential equations

- $\mathrm{P} = \text{population}, \mathrm{t} = \text{time(days)}$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$
- $|P| = e^{Kt+C_1} = e^{Kt}e^{C_1}$

Modeling with differential equations

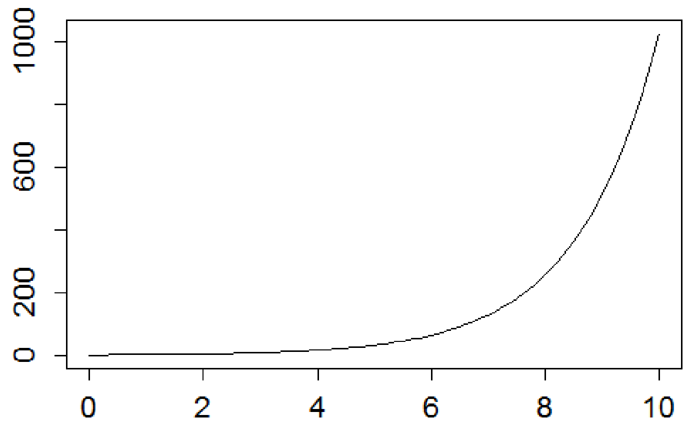
- $\mathrm{P} = \text{population}, \mathrm{t} = \text{time(days)}$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$
- $|P| = e^{Kt+C_1} = e^{Kt}e^{C_1} = Ce^{Kt}$

Exponential Growth

- P = population, t = time(days)
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$
- $|P| = e^{Kt+C_1} = e^{Kt}e^{C_1} = Ce^{Kt}$

$$\boxed{P(t) = Ce^{Kt}}$$

SOLUTION



Verification

$$P'(t) = (Ce^{Kt})'$$

Verification

$$P'(t) = (Ce^{Kt})' = KCe^{Kt}$$

Verification

$$P'(t) = (Ce^{Kt})' = KCe^{Kt} = KP(t)$$

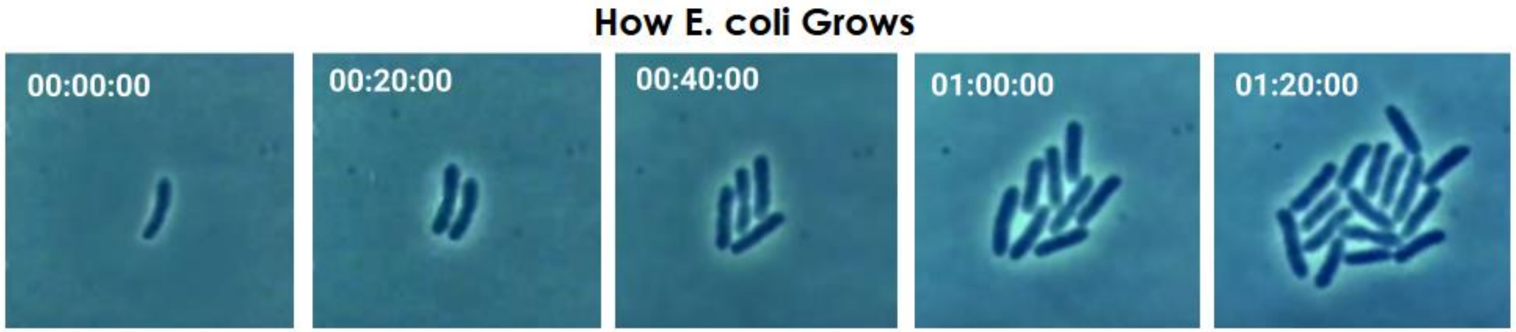
Verification

$$P'(t) = (Ce^{Kt})' = KCe^{Kt} = KP(t) = KP$$

Example

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \qquad P = 1$
- $t = 80 \quad P = 16$



Example

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \qquad P = 1 \qquad 1 = Ce^{0}$
- $t = 80 \quad P = 16$

Example

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \qquad P = 1 \qquad 1 = Ce^0 = C$
- $t = 80 \quad P = 16$

Example

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \qquad P = 1 \qquad 1 = Ce^{0} = C$
- $t = 80 \quad P = 16 \quad 16 = e^{80K}$

Example

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \quad P = 1 \quad 1 = Ce^0 = C$
- $t = 80 \quad P = 16 \quad 16 = e^{80K}$
- $ln16 = 80K$

Example

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \qquad P = 1 \qquad 1 = Ce^0 = C$
- $t = 80 \qquad P = 16 \qquad 16 = e^{80K}$
- $ln16 = 80K$
- $K = \frac{ln16}{80}$

Example

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \qquad P = 1 \qquad 1 = Ce^{0} = C$
- $t = 80 \qquad P = 16 \qquad 16 = e^{80K}$
- $ln16 = 80K$
- $K = \frac{ln16}{80}$

$$\boxed{P(t) = e^{\frac{ln16}{80}t}}$$

Example

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 1 \qquad 1 = Ce^{0} = C$
- $t = 80 \qquad P = 16 \qquad 16 = e^{80K}$
- $ln16 = 80K$
- $K = \frac{ln16}{80}$

$$P(t) = e^{\frac{ln16}{80}t}$$

C → Population at time 0

Example

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \qquad P = 1 \qquad 1 = Ce^{0} = C$
- $t = 80 \qquad P = 16 \qquad 16 = e^{80K}$
- $ln16 = 80K$
- $K = \frac{ln16}{80}$

$$\boxed{P(t) = e^{\frac{ln16}{80}t}}$$

C → Population at time 0

K → coefficient of growth

Example

$$\boxed{P = Ce^{Kt}}$$

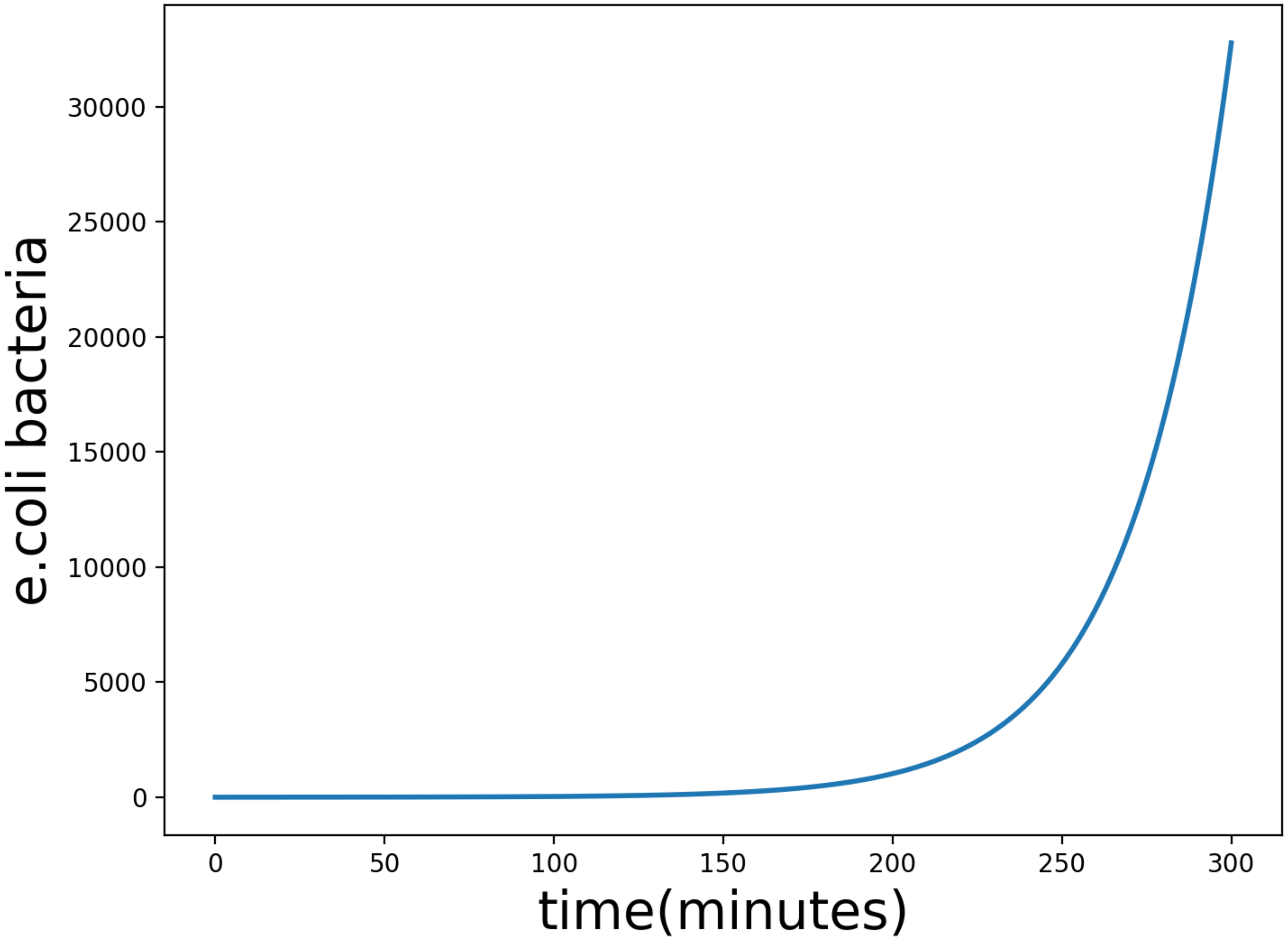
- $t = 0 \qquad P = 1 \qquad 1 = Ce^{0} = C$
- $t = 80 \qquad P = 16 \qquad 16 = e^{80K}$
- $ln16 = 80K$
- $K = \frac{ln16}{80}$

$$\boxed{P(t) = e^{\frac{ln16}{80}t}}$$

C → Population at time 0

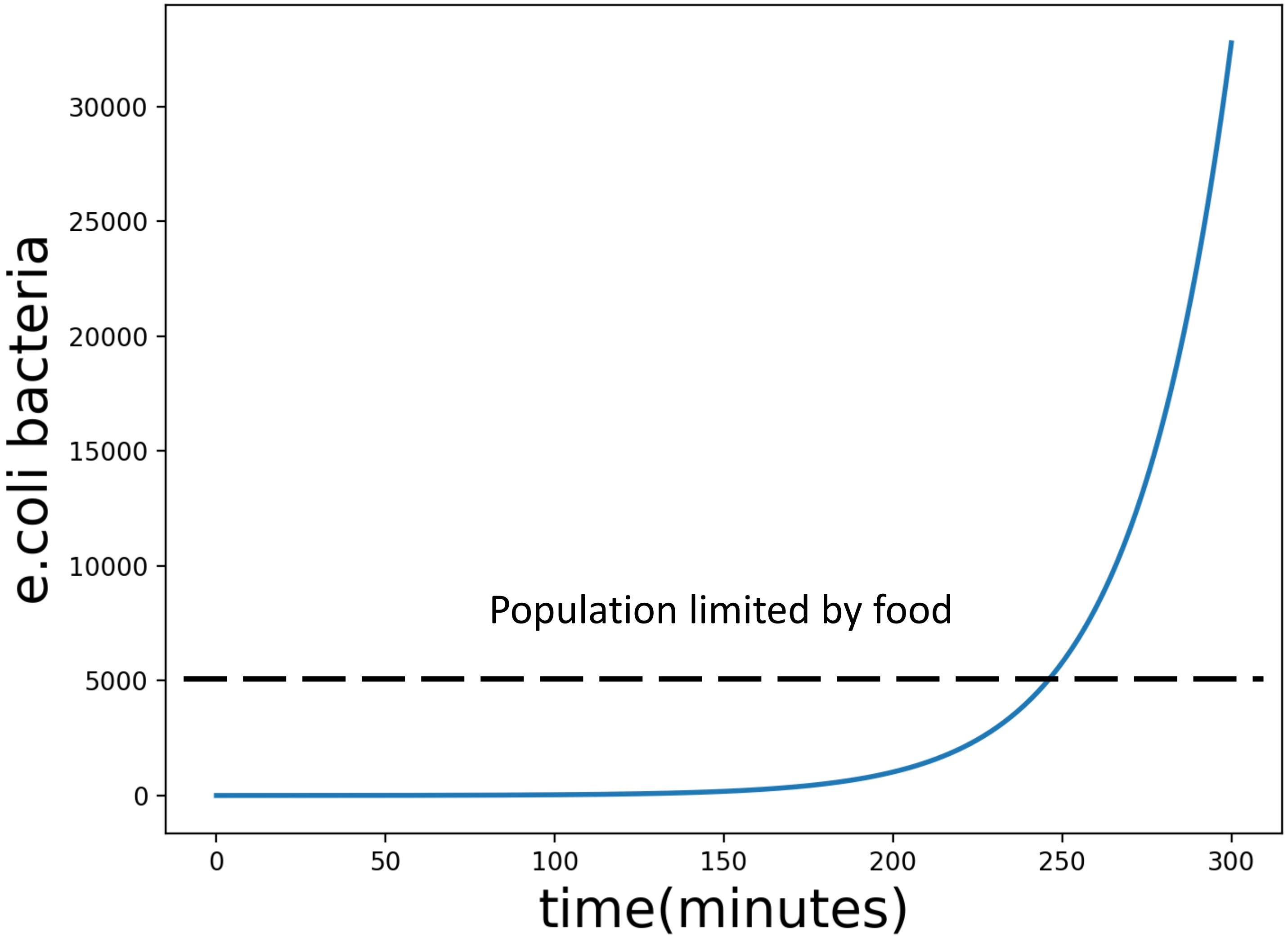
K → coefficient of growth (unit is 1/t)

Mathematical Model of e.coli Growth



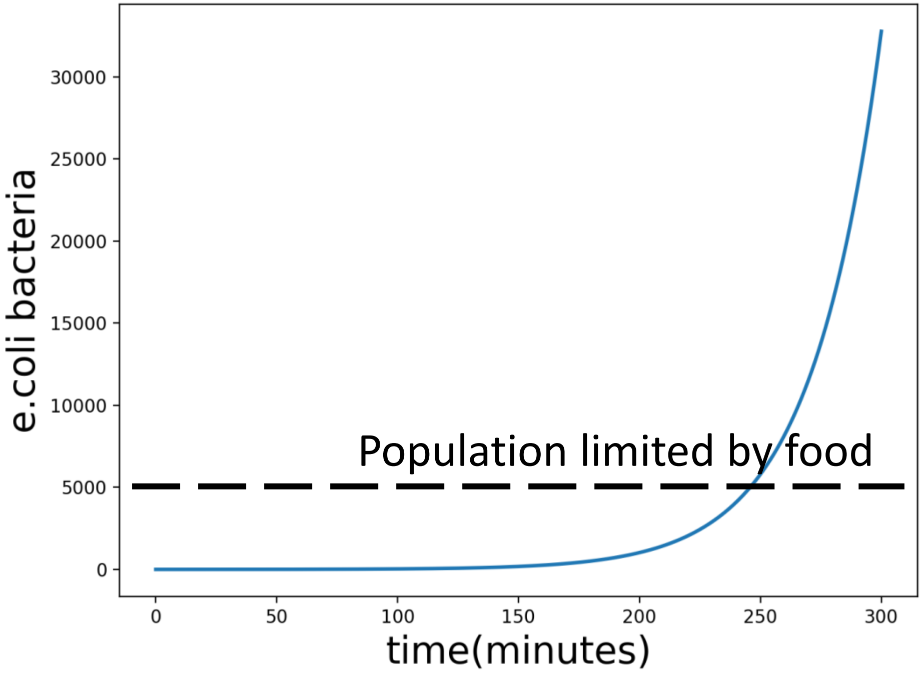
*also used to compute compound interest rates

Population growth N relative to food C



Population growth N relative to food C

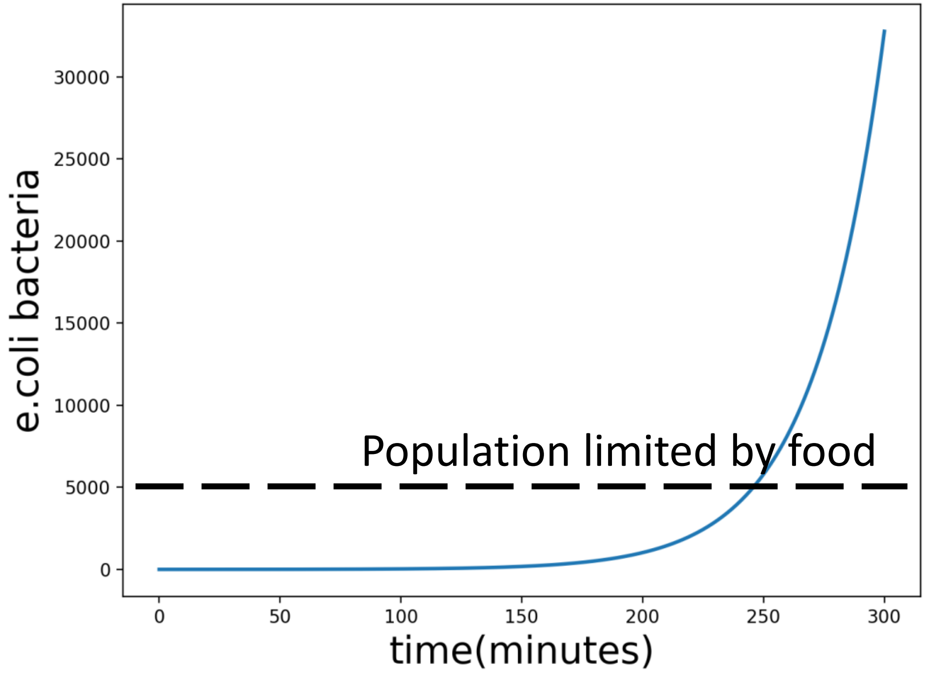
- $N' = K(C)N$



Population growth N relative to food C

- $N' = K(C)N$

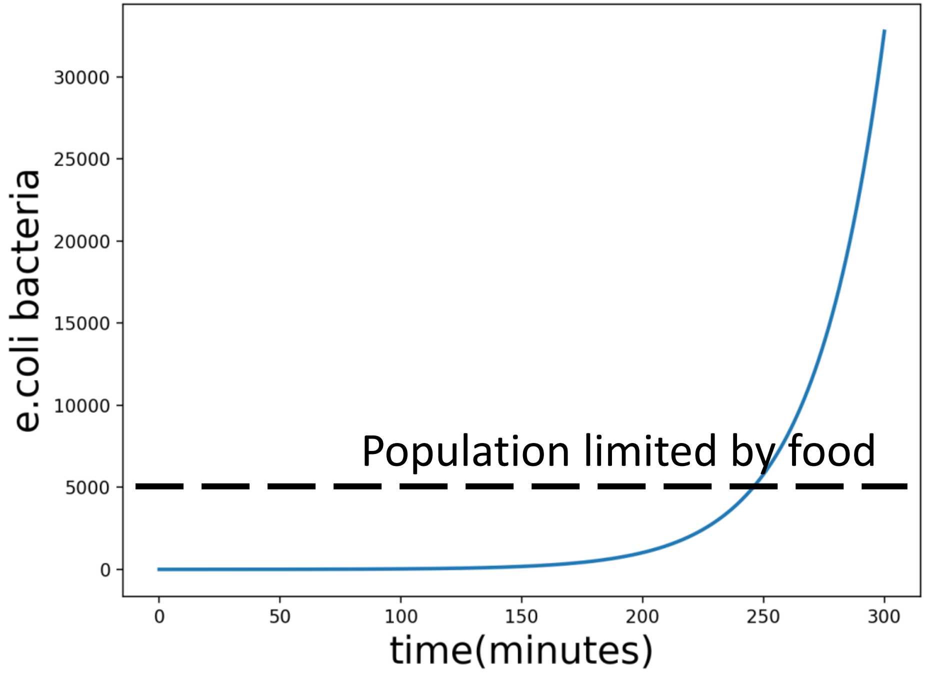
K is a function of how food is converted to growth



Population growth N relative to food C

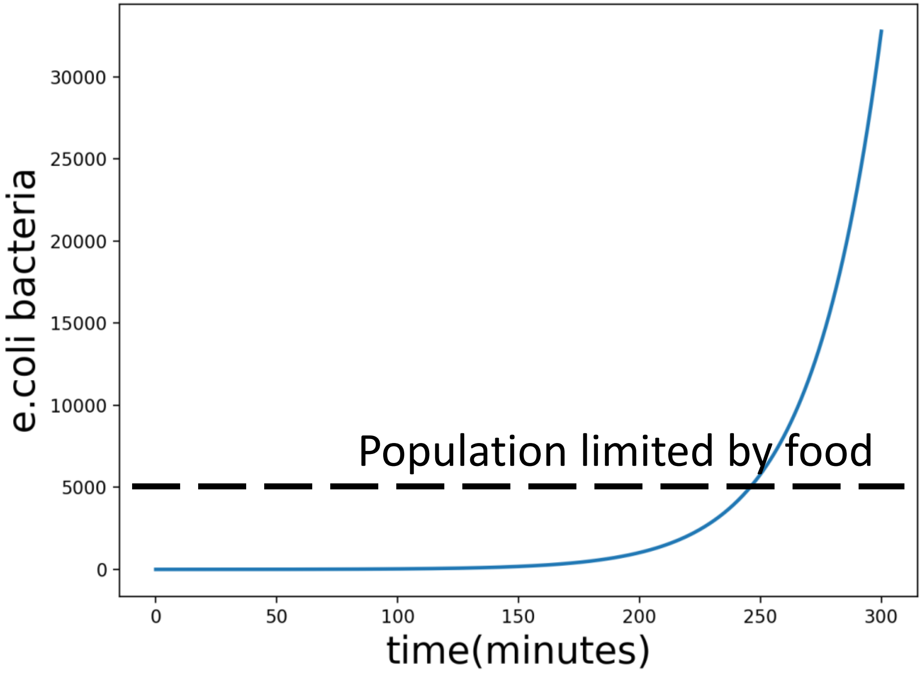
- $N' = K(C)N$

C(t) is food concentration



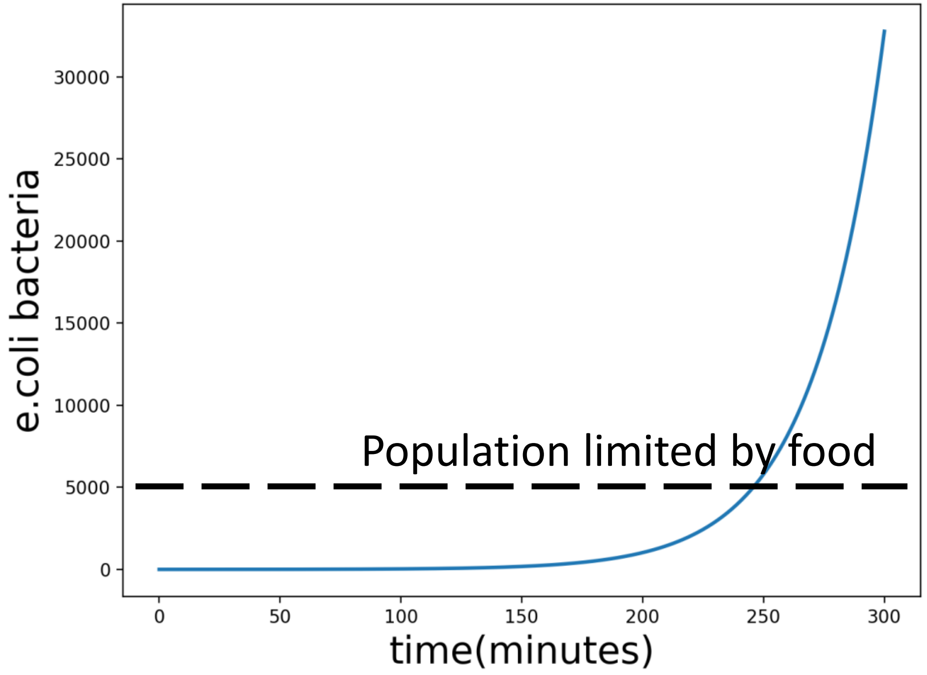
Population growth N relative to food C

- $N' = K(C)N$
- $C' =$



Population growth N relative to food C

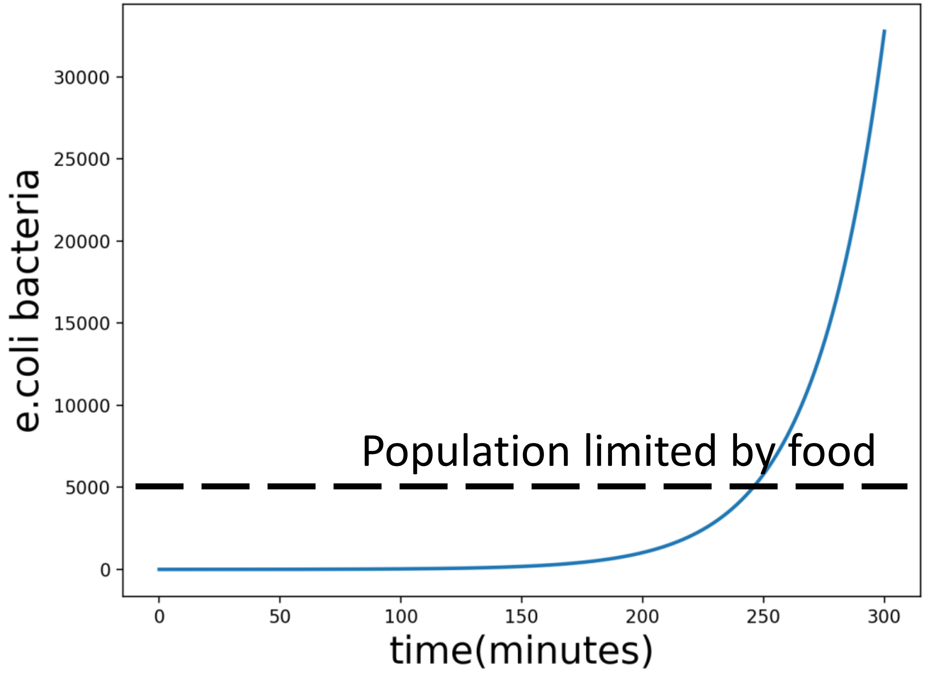
- $N' = K(C)N$
- $C' = -\alpha N'$



Population growth N relative to food C

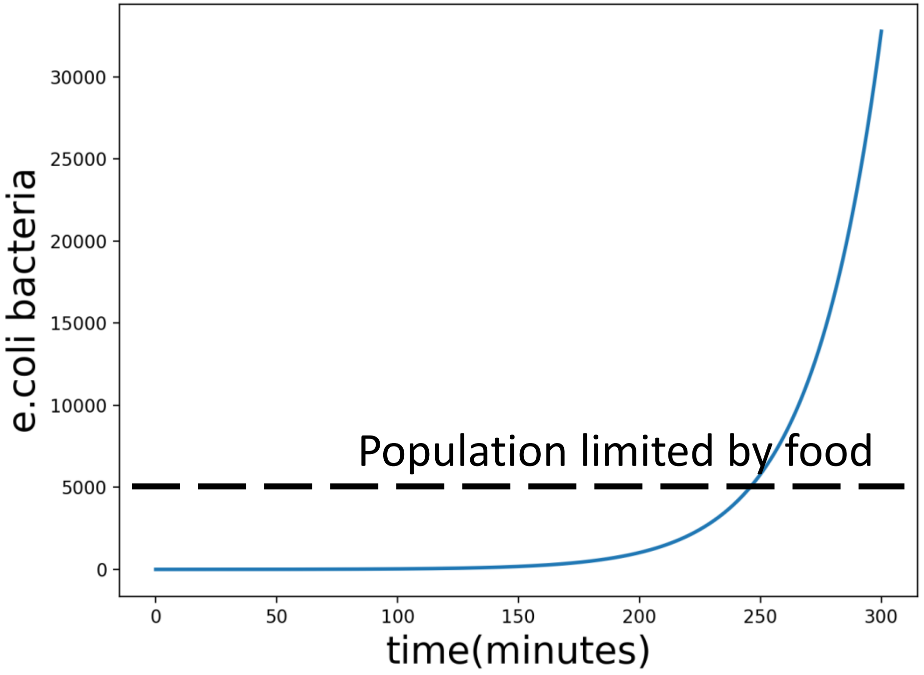
- $N' = K(C)N$
- $C' = -\alpha N'$

α: constant describing consumption of food by each new bacterium



Population growth N relative to food C

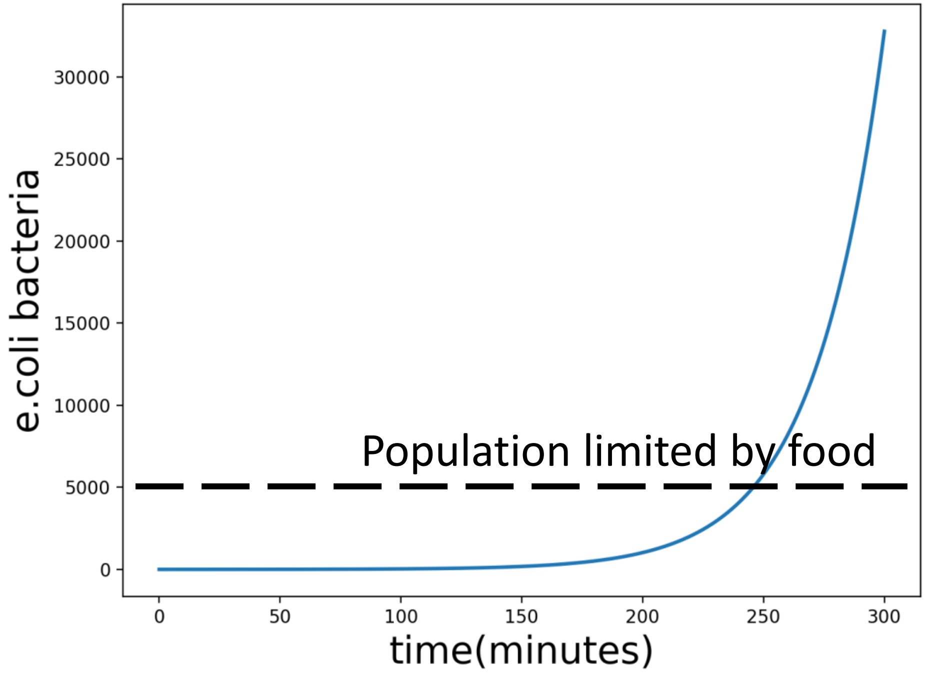
- $N' = K(C)N$
- $C' = -\alpha K(C)N$



Population growth N relative to food C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

If $K(C) = kC$

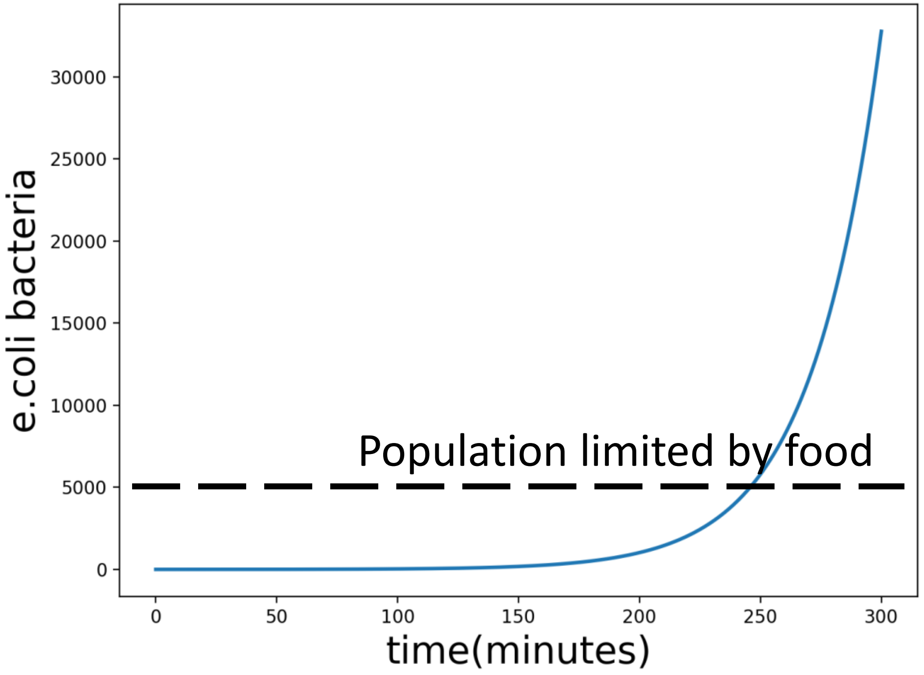


Population growth N relative to food C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

If $K(C) = kC$

- $C' = -\alpha N'$

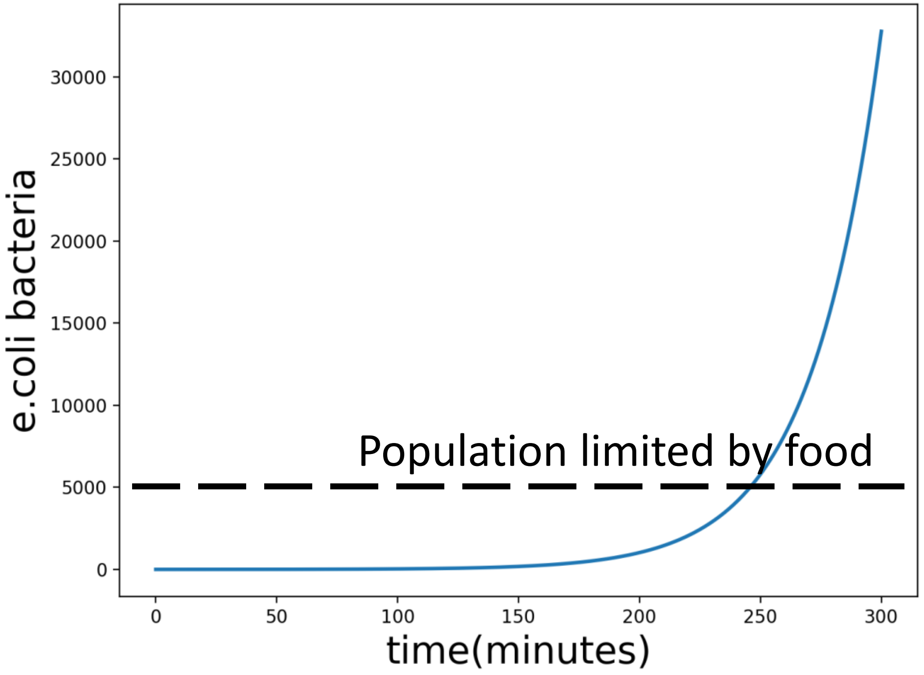


Population growth N relative to food C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

If $K(C) = kC$

- $C' = -\alpha N'$
- $C = -\alpha N + C_0$

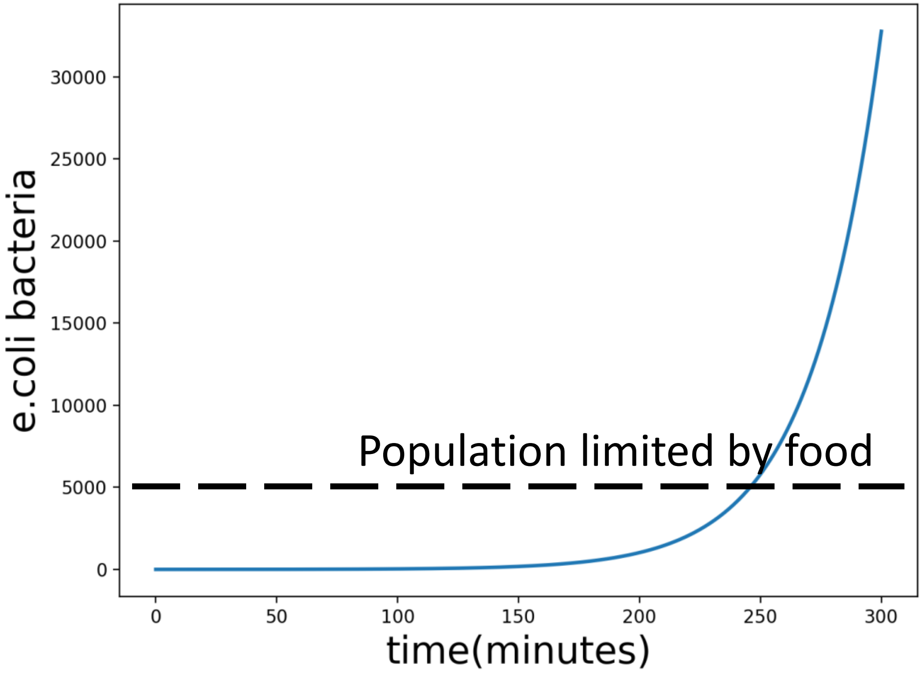


Population growth N relative to food C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

If $K(C) = kC$

- $C' = -\alpha N'$
- $C = -\alpha N + C_0$
- $N' = k(C_0 - \alpha N)N$

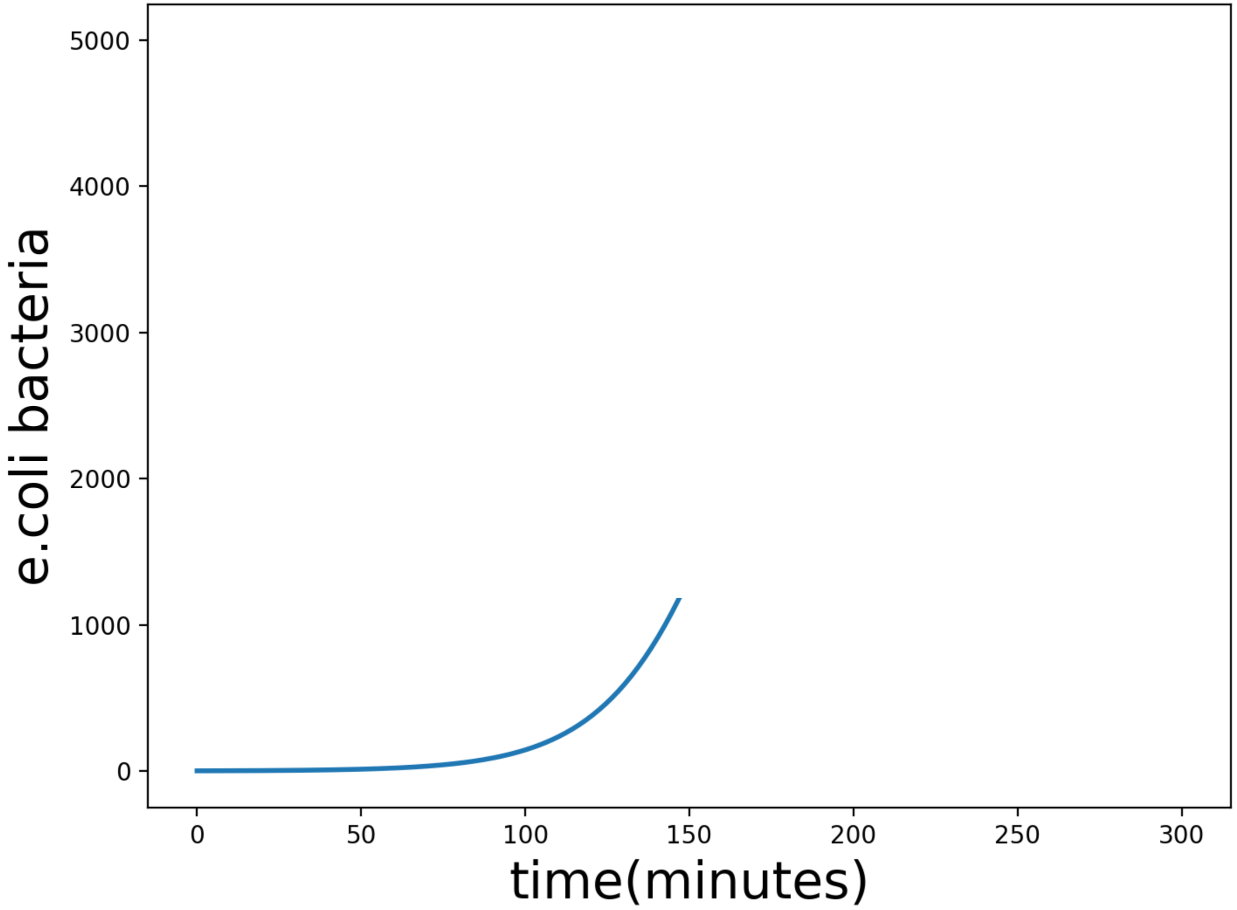


Population growth N relative to food C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

If $K(C) = kC$

- $C' = -\alpha N'$
- $C = -\alpha N + C_0$
- $N' = k(C_0 - \alpha N)N$

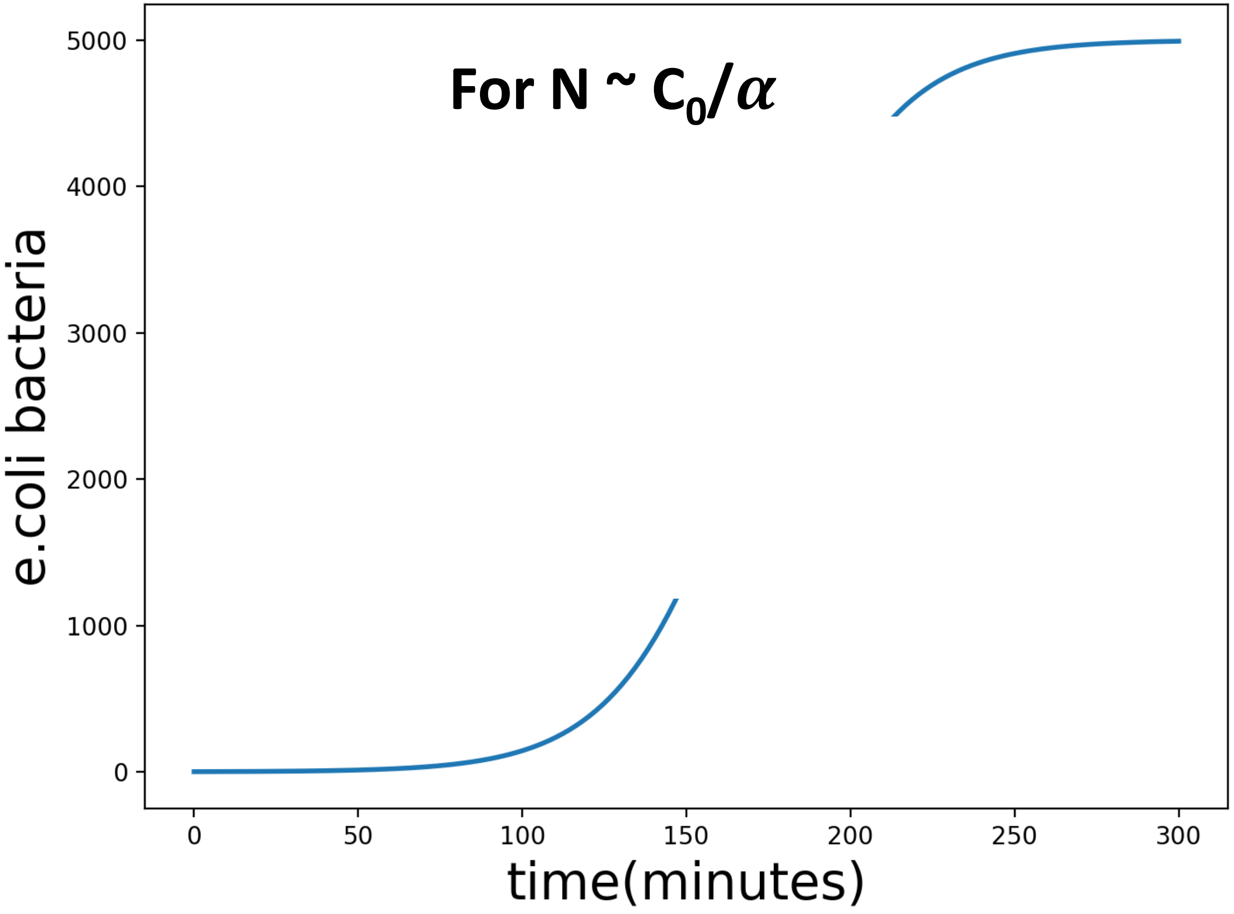


Population growth N relative to food C

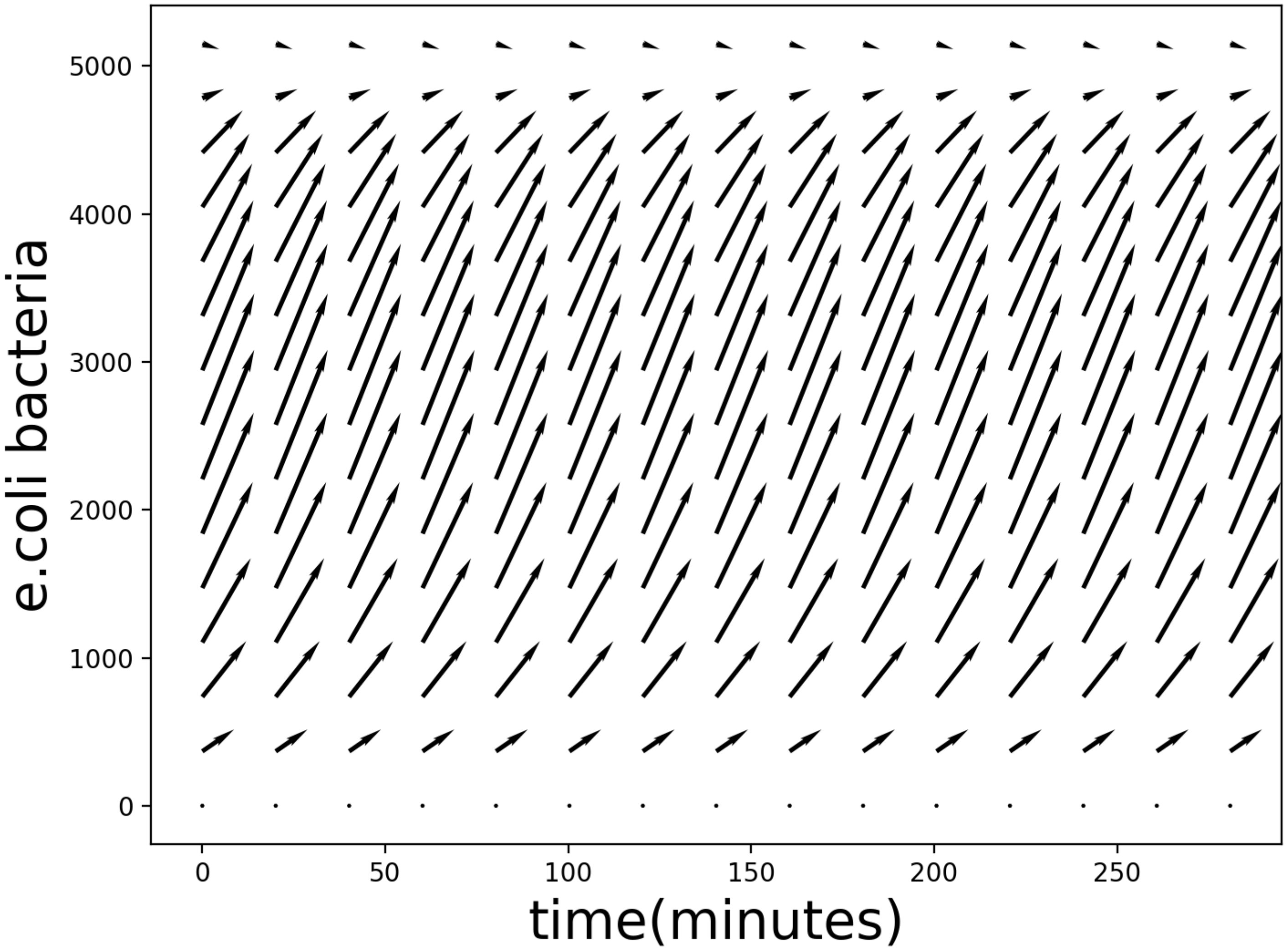
- $N' = K(C)N$
- $C' = -\alpha K(C)N$

If $K(C) = kC$

- $C' = -\alpha N'$
- $C = -\alpha N + C_0$
- $N' = k(C_0 - \alpha N)N$



Vector Field

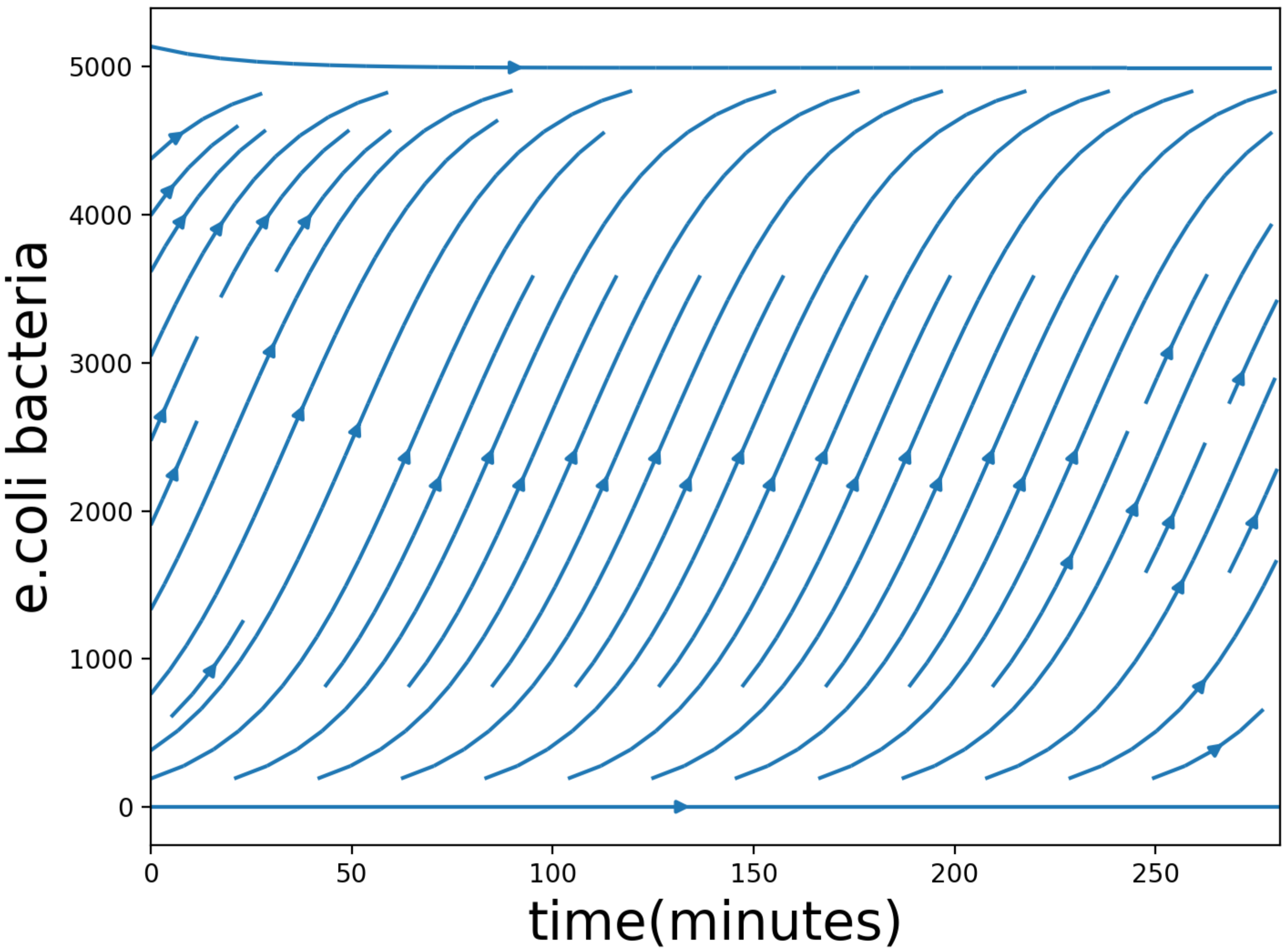


matplotlib.pyplot.quiver

`matplotlib.pyplot.quiver(\*args, data=None, \*\*kw)` [source]

```
X = np.linspace(0, 300, 300)
Y = np.linspace(0 ,5000, 300)
X, Y = np.meshgrid(X, Y)

dxdt = # some differential equation e.g. X**2.0 * 0.0001
dt = scale * np.ones(X.shape)
dx = dxdt * dt

plt.quiver(X[::20,::20], Y[::20, ::20], dt[::20, ::20],
dx[::20,::20], headwidth=2.0, angles='xy', scale=25.)
```

Stream Plot



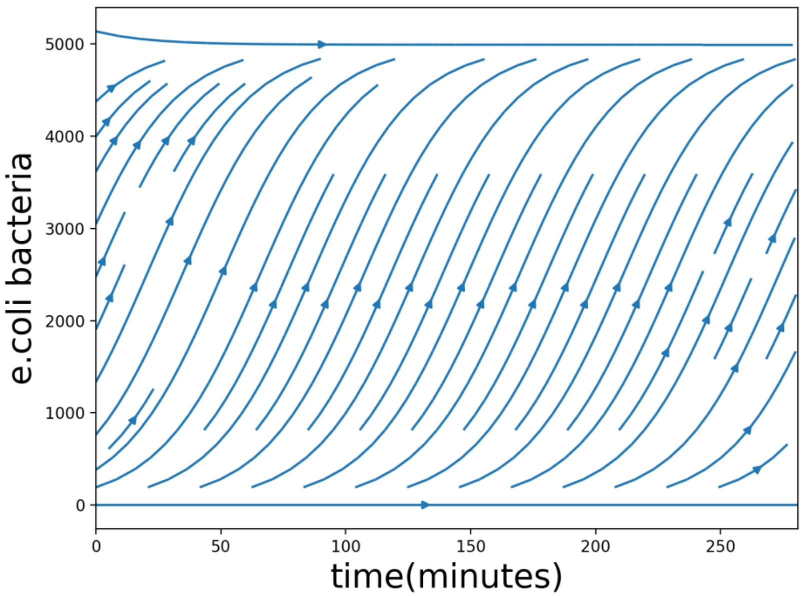
5000
4000
3000
2000
1000
0
e.coli bacteria
0
50
100
150
200
250
time(minutes)

matplotlib.pyplot.streamplot

```
matplotlib.pyplot.streamplot(x, y, u, v, density=1, linewidth=None, color=None, cmap=None,
norm=None, arrowsize=1, arrowstyle='-|>', minlength=0.1, transform=None, zorder=None,
start_points=None, maxlength=4.0, integration_direction='both', *, data=None)
```

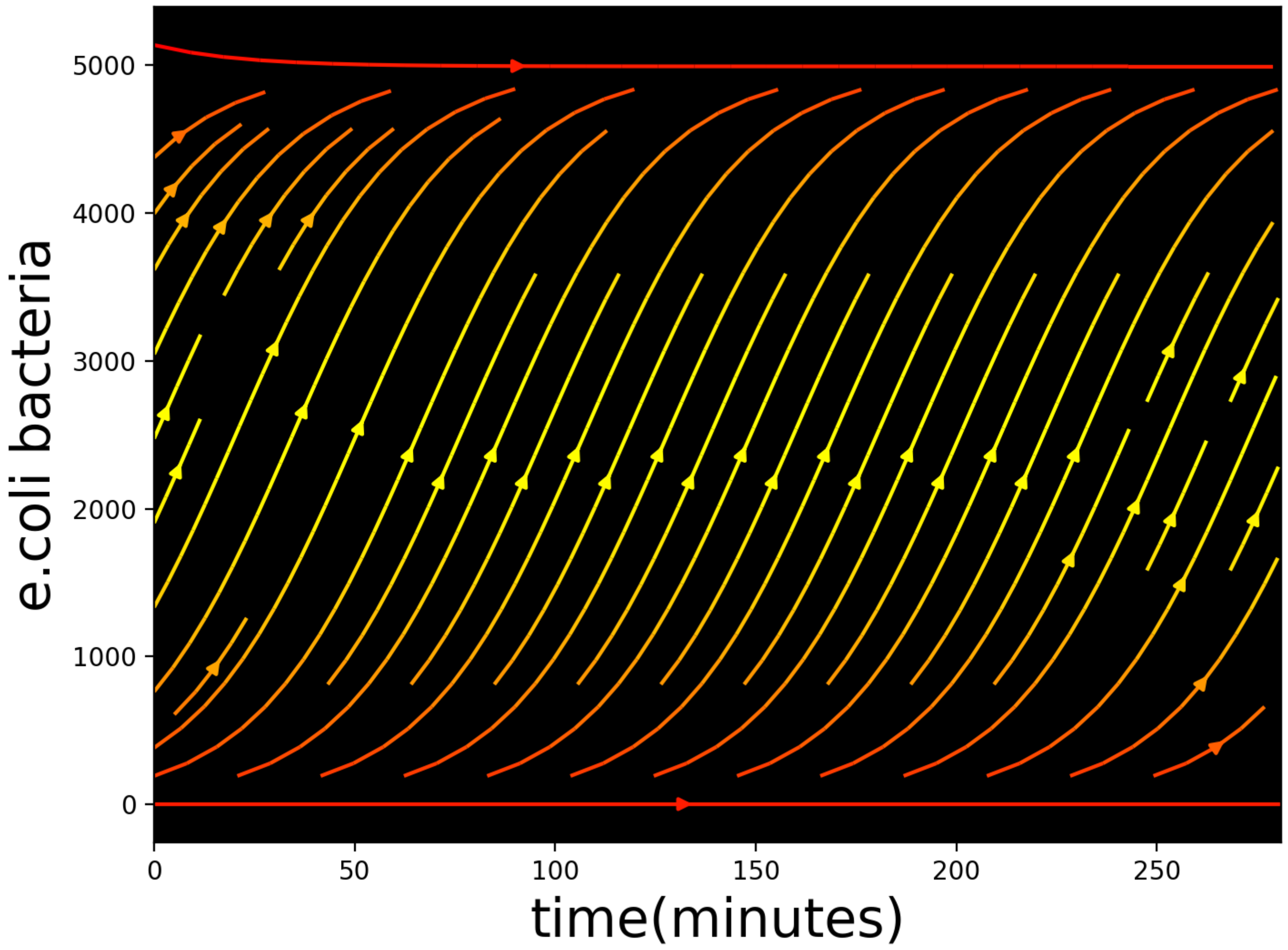
[source]

```
# … analogous to quiver

plt.streamplot(X, Y, dt, dx)
```



Add color

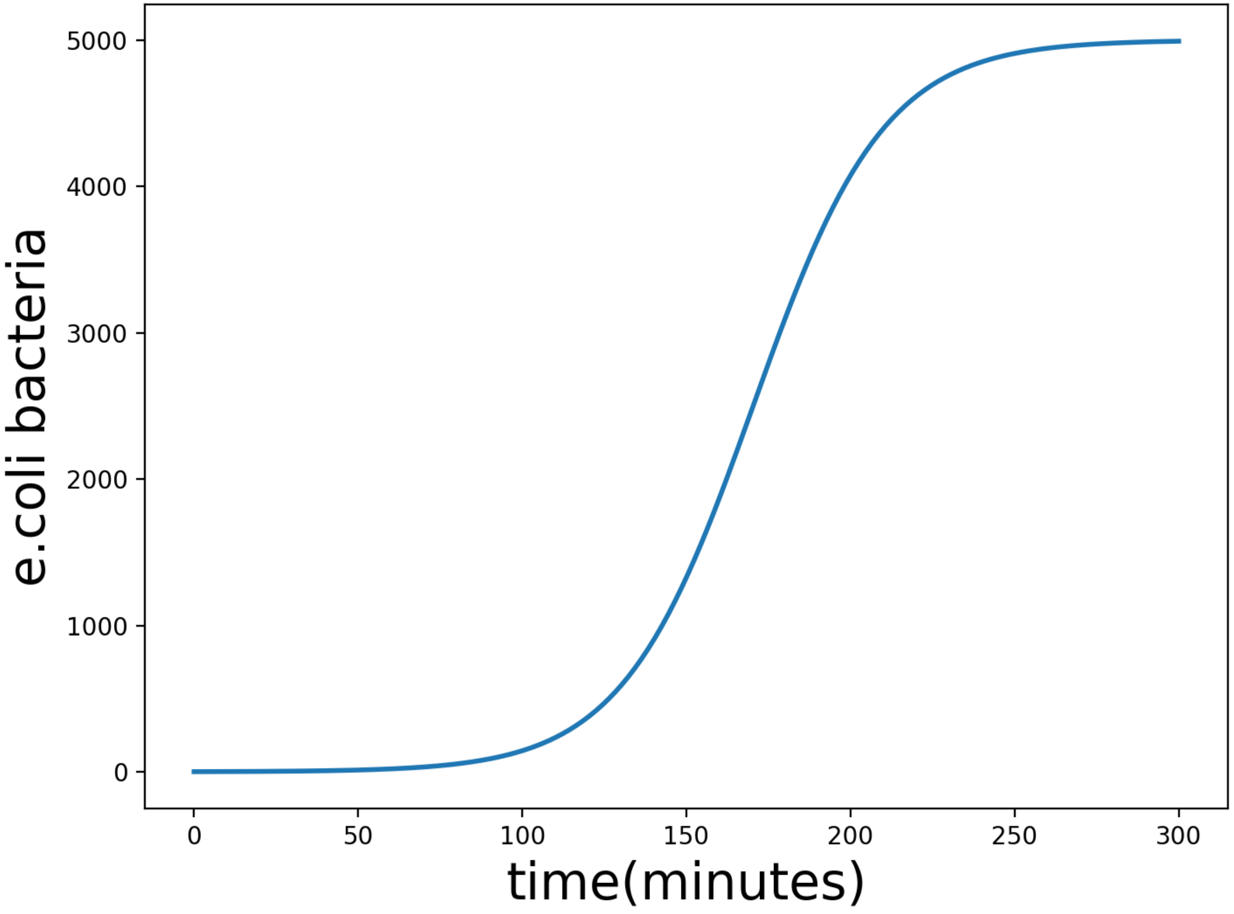
```
#in streamplot…
color = dx
```



Logistic Growth Function

$$N(t) = \frac{N_0 B}{N_0 + (B - N_0)e^{-rt}}$$

$$N_0 = N(0), B = \frac{C_0}{\alpha}, r = kC_0$$



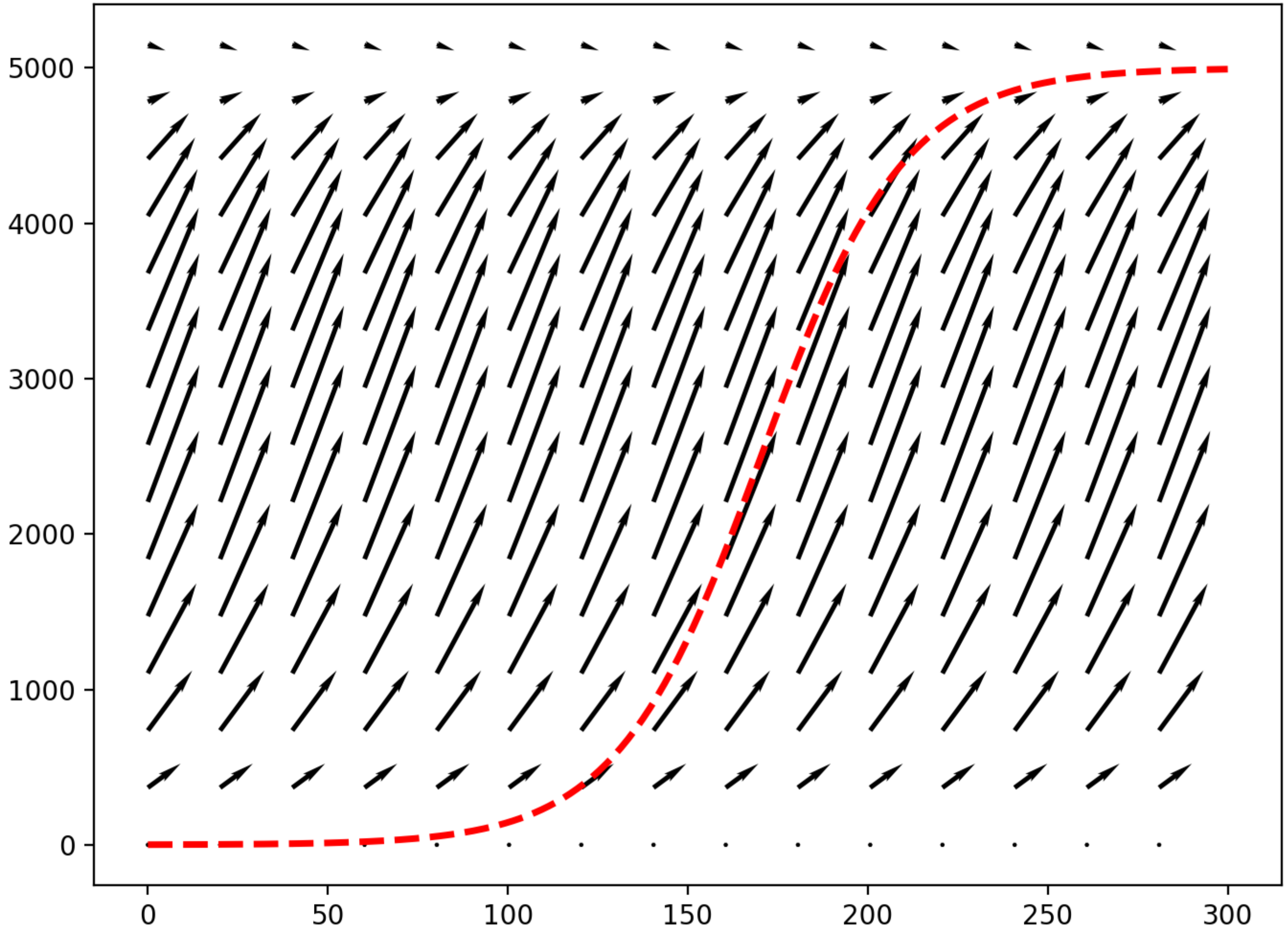
Numerical solutions

- There is usually **no closed form solution** for a system of differential equations, unless the problem is really simple
- We look therefore for an **approximate** solution

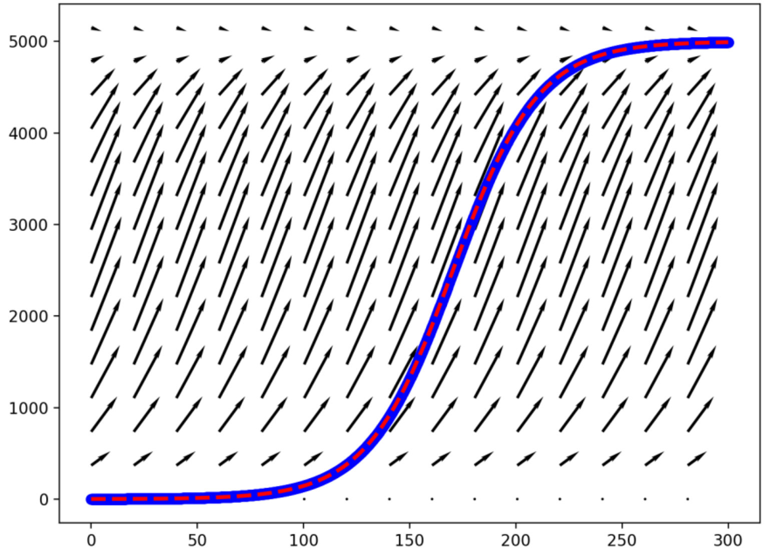
Field of derivatives

- We know how to compute f
- This means that we can compute the **vector/slope field** of x

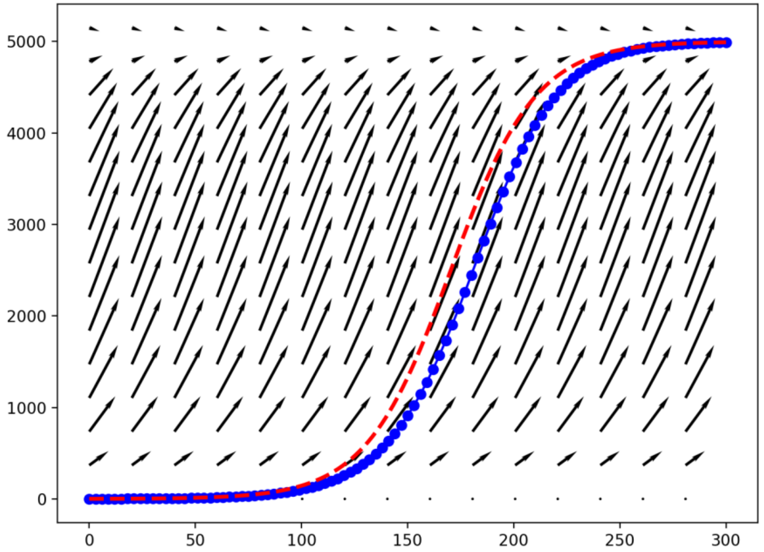
Vector/Slope field



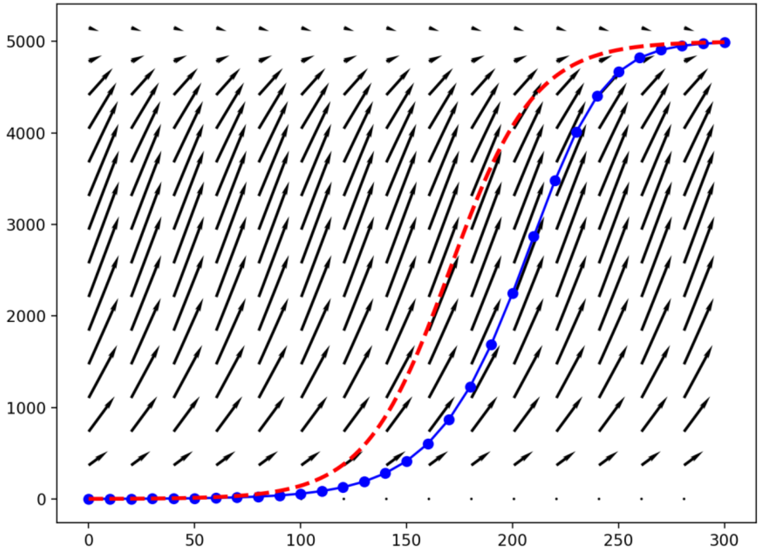
Forward Euler Numerical Solution



dt = 0.1



dt = 3



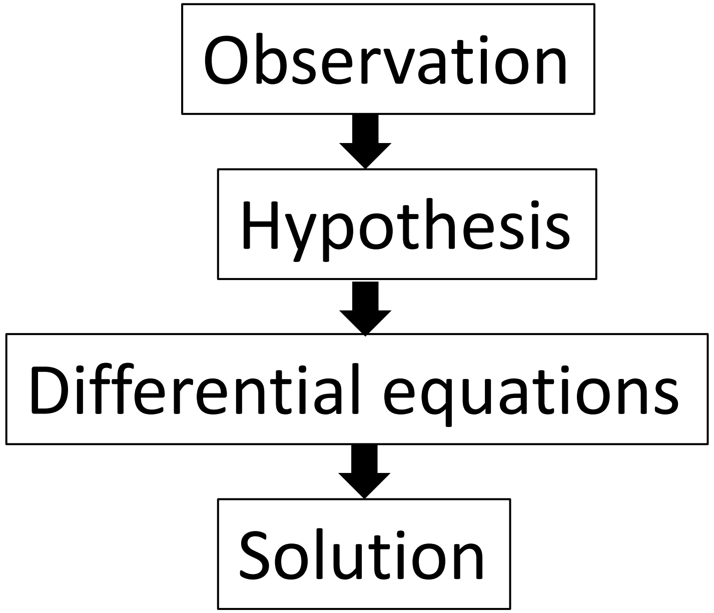
dt = 10

Follow the vector field: `x[n+1] = x[n] + dt*f(x[n], t[n])`

Python Implementation

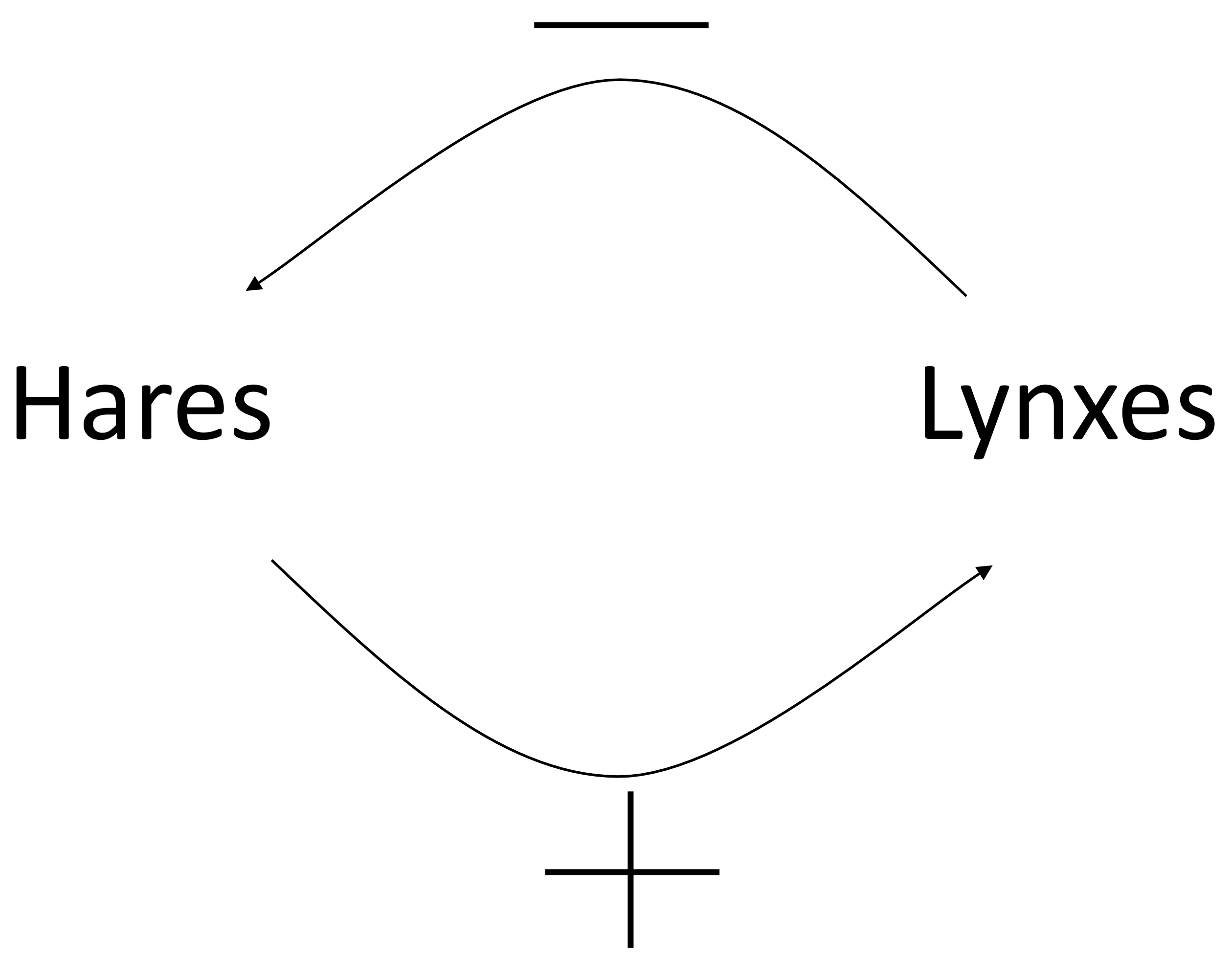
```
# f is your differential equation

def ode_FE(f, X_0, dt, T):
    steps = int(round(float(T)/dt))
    x = np.zeros(steps + 1)
    t = np.linspace(0, steps * dt, len(x))
    x[0] = X_0
    for n in range(steps):
        x[n+1] = x[n] + dt*f(x[n])
    return x, t
```

Modeling Process



Hares and Lynxes Population Growth Model

Predator – Prey Relation



Hares and Lynxes Population Growth Model

- Population y (lynxes) changes in time due to reproduction limited by population x (hares)

Hares and Lynxes Population Growth Model

- Population y (lynxes) changes in time due to reproduction limited by population x (hares)

$$\frac{dy}{dt} = dx$$

Hares and Lynxes Population Growth Model

- Population y (lynxes) changes in time due to reproduction limited by population x (hares)

$$\frac{dy}{dt} = dxy$$

Hares and Lynxes Population Growth Model

- Population y (lynxes) changes in time due to reproduction limited by population x (hares)
- Some die due to age

$$\frac{dy}{dt} = dxy$$

Hares and Lynxes Population Growth Model

- Population y (lynxes) changes in time due to reproduction limited by population x (hares)
- Some die due to age

$$\frac{dy}{dt} = dxy - cy$$

Hares and Lynxes Population Growth Model

Lynxes: $\frac{dy}{dt} = dxy \; - cy$

Hares: $\frac{dx}{dt} =$

Hares and Lynxes Population Growth Model

Lynxes: $\frac{dy}{dt} = dxy\ - cy$

Hares: $\frac{dx}{dt} = ax$

Hares and Lynxes Population Growth Model

Lynxes: $\frac{dy}{dt} = dxy \ - cy$

Hares: $\frac{dx}{dt} = ax - bxy$

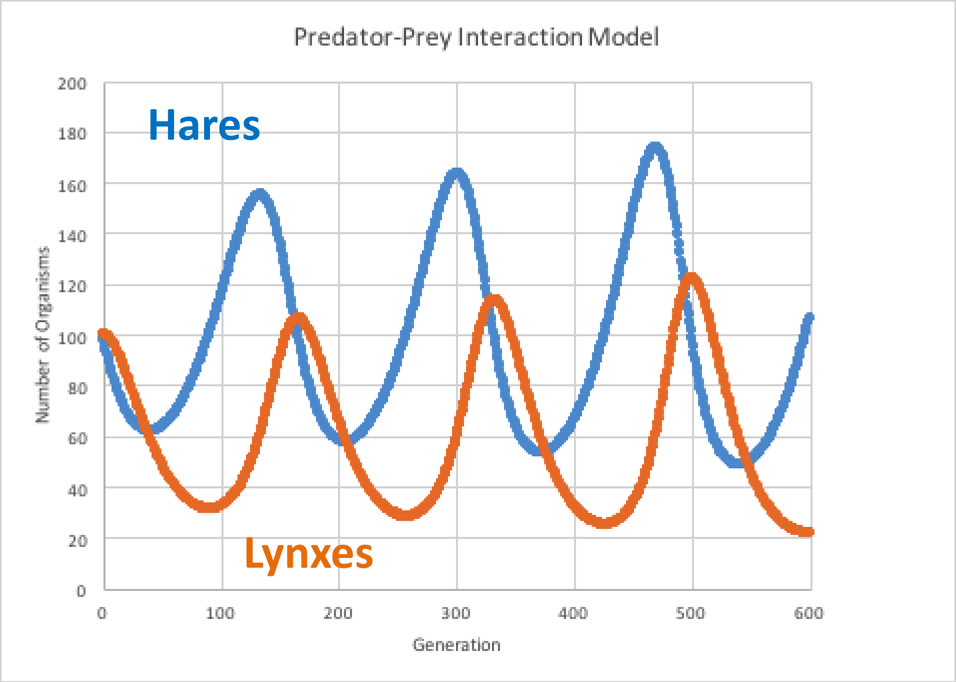
Hares and Lynxes Population Growth Model

Lynxes: $\frac{dy}{dt} = dxy \ - cy$

Hares: $\frac{dx}{dt} = ax - bxy$

Multiply by lynxes to reflect effects of predation

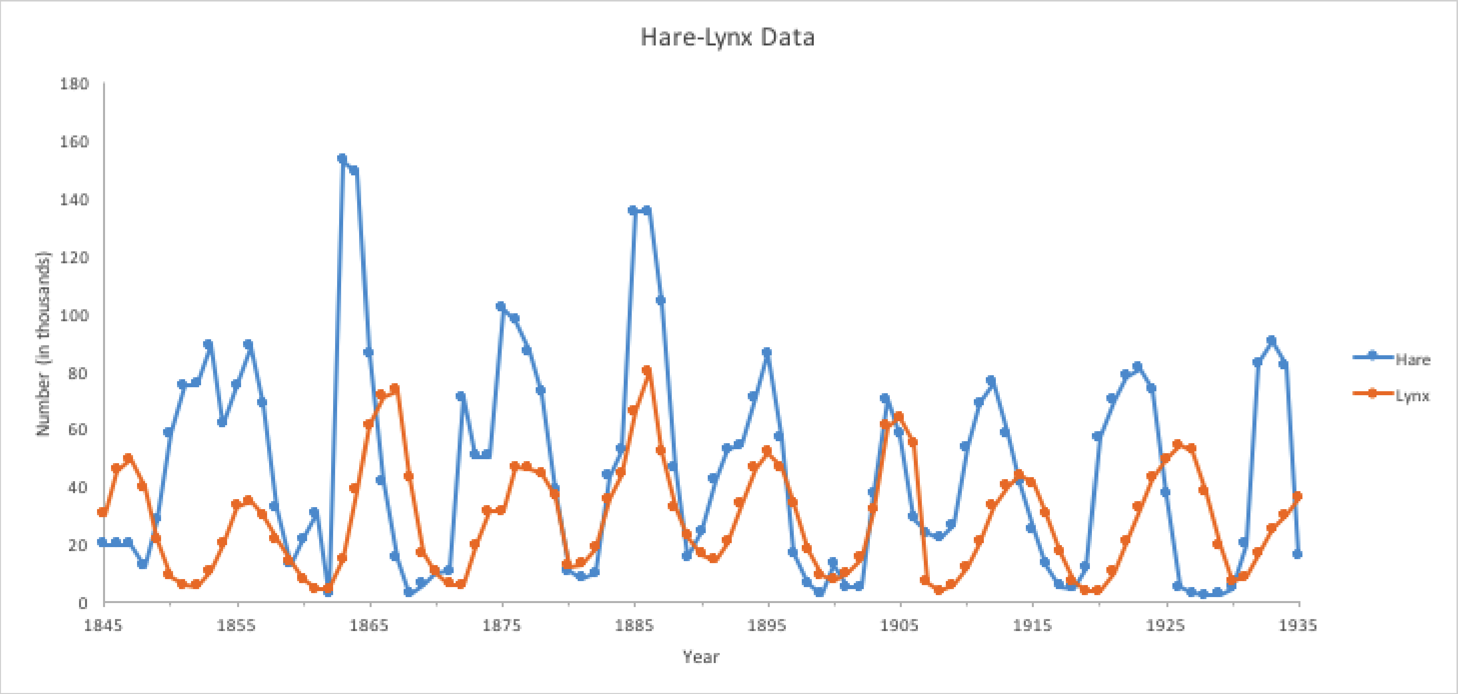
Example Model Solution



Lynxes: $\frac{dy}{dt} = dxy \ - cy$

Hares: $\frac{dx}{dt} = ax - bxy$

Real World


Hare-Lynx Data
180
160
140
120
100
80
60
40
20
0
Number (in thousands)
1845
1855
1865
1875
1885
1895
1905
1915
1925
1935
Year
Hare
Lynx

Lotka-Volterra Predator-Prey Model

Exercise

- Plot a vector field and a stream plot for the exponential growth model.
- At the start of an experiment there are 10 plants of Wolffia microscopica. At time = 240 hours there are 1015. How many plants will there be at 20 days? Model the population growth assuming a constant exponential growth rate and plot the graph using logarithmic scale.

Exercise

- Plot a vector field and a stream plot for the logistic growth model. Express the rate of change of population size with a color map.
- Bacteria grow at a rate of 20% per hour in a petri dish. If there is initially one bacterium and a carrying capacity of 1 million cells, how long does it take to reach 500,000 cells? Plot the graph using logarithmic scale.

Exercise (for next week)

- Assume the following system of differential equations, where S is the number of susceptible, I the number of infected and R the number of recovered people in a population suffering from a virus outbreak. N is the sum of S, I and R. Plot the graphs for the variables S,I, and R assuming beta = 0.5 and gamma = 0.1 using the Forward Euler method to numerically solve the system of ODE for a time period of 200 days. Assume that at the start of the outbreak we have 1000 susceptible and 1 infected person. At which day do we have the greatest number of infected? Explain what the coefficients beta and gamma represent in the context of a virus epidemic. Why is N set equal to the sum of the three variables?

$$\frac{dS}{dt} = -\frac{\beta I S}{N},$$

$$\frac{dI}{dt} = \frac{\beta I S}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I,$$