SPB 3

dr Wojtek Palubicki

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} =$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$
- $|P| = e^{Kt+C_1}$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$
- $|P| = e^{Kt+C_1} = e^{Kt}e^{C_1}$

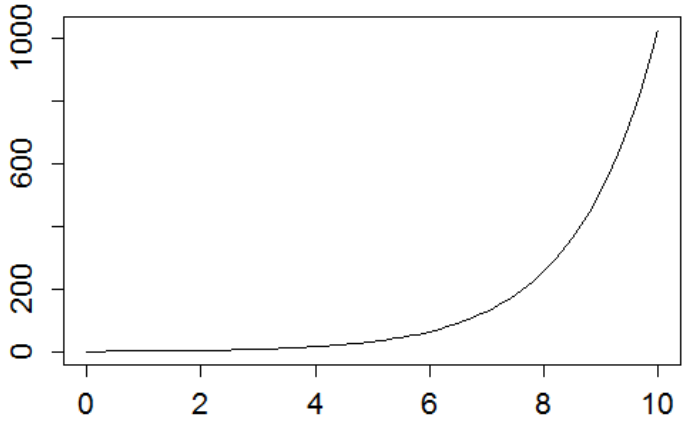
Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$
- $|P| = e^{Kt+C_1} = e^{Kt}e^{C_1} = Ce^{Kt}$

Modelowanie za pomocy równań różniczkowych

- $P = populacja\ t = czas\ (dniach)$
- $\frac{dP}{dt} = KP$
- $\frac{1}{P}\frac{dP}{dt} = K$
- $\frac{1}{P}dP = Kdt$
- $\int \frac{1}{P}dP = \int Kdt$
- $ln|P| = Kt + C_1$
- $|P| = e^{Kt+C_1} = e^{Kt}e^{C_1} = Ce^{Kt}$

$$P = Ce^{Kt}$$



Przykład

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 100$
- $t = 50 \qquad P = 200$

Przykład

$$\boxed{P = Ce^{Kt}}$$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^{0}$
- $t = 50 \qquad P = 200$

Przykład

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^{0} = C$
- $t = 50 \qquad P = 200$

Przykład

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^{0} = C$
- $t = 50 \qquad P = 200 \qquad 200 = 100e^{50K}$

Przykład

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^{0} = C$
- $t = 50 \qquad P = 200 \qquad 200 = 100e^{50K}$
- $2 = e^{50K}$

Przykład

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^{0} = C$
- $t = 50 \qquad P = 200 \qquad 200 = 100e^{50K}$
- $2 = e^{50K}$
- $ln2 = 50K$

Przykład

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^{0} = C$
- $t = 50 \qquad P = 200 \qquad 200 = 100e^{50K}$
- $2 = e^{50K}$
- $ln2 = 50K$
- $K = \frac{ln2}{50}$

Przykład

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^0 = C$
- $t = 50 \qquad P = 200 \qquad 200 = 100e^{50K}$
- $2 = e^{50K}$
- $ln2 = 50K$
- $K = \frac{ln2}{50}$

$$P(t) = 100e^{\frac{ln2}{50}t}$$

Przykład

$P = Ce^{Kt}$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^0 = C$
- $t = 50 \qquad P = 200 \qquad 200 = 100e^{50K}$
- $2 = e^{50K}$
- $ln2 = 50K$
- $K = \frac{ln2}{50}$

C → Populacja w czasie 0

$P(t) = 100e^{\frac{ln2}{50}t}$

Przykład

$$P = Ce^{Kt}$$

- $t = 0 \qquad P = 100 \qquad 100 = Ce^0 = C$
- $t = 50 \qquad P = 200 \qquad 200 = 100e^{50K}$
- $2 = e^{50K}$
- $ln2 = 50K$
- $K = \frac{ln2}{50}$

$$P(t) = 100e^{\frac{ln2}{50}t}$$

C → Populacja w czasie 0

K → współczynnik wzrostu

Prawo Newtona Ochłodzenia

- $\frac{dT}{dt} = -K(T - T_a)$

Prawo Newtona Ochłodzenia

- $\frac{dT}{dt} = -K(T - T_a)$
- $\frac{1}{T-T_a} dT = -Kdt$

Prawo Newtona Ochłodzenia

- $\frac{dT}{dt} = -K(T - T_a)$
- $\frac{1}{T-T_a} dT = -Kdt$
- $\int \frac{1}{T-T_a} dT = \int -Kdt$

Prawo Newtona Ochłodzenia

- $\frac{dT}{dt} = -K(T - T_a)$
- $\frac{1}{T-T_a} dT = -Kdt$
- $\int \frac{1}{T-T_a} dT = \int -Kdt$
- $ln|T - T_a| = -Kt + C_1$

Prawo Newtona Ochłodzenia

- $\frac{dT}{dt} = -K(T - T_a)$
- $\frac{1}{T-T_a} dT = -Kdt$
- $\int \frac{1}{T-T_a} dT = \int -Kdt$
- $ln|T - T_a| = -Kt + C_1$
- $|T - T_a| = e^{-Kt+C_1} = Ce^{-Kt}$

Prawo Newtona Ochłodzenia

- $\frac{dT}{dt} = -K(T - T_a)$
- $\frac{1}{T-T_a} dT = -Kdt$
- $\int \frac{1}{T-T_a} dT = \int -Kdt$
- $ln|T - T_a| = -Kt + C_1$
- $|T - T_a| = e^{-Kt+C_1} = Ce^{-Kt}$
- $T \geq T_a$ $\qquad\qquad T(\mathrm{t}) = Ce^{-Kt} + T_a$

Prawo Newtona Ochłodzenia

- $\frac{dT}{dt} = -K(T - T_a)$
- $\frac{1}{T-T_a} dT = -Kdt$
- $\int \frac{1}{T-T_a} dT = \int -Kdt$
- $ln|T - T_a| = -Kt + C_1$
- $|T - T_a| = e^{-Kt+C_1} = Ce^{-Kt}$
- $T \geq T_a \qquad T(\mathrm{t}) = Ce^{-Kt} + T_a$
- $T < T_a \qquad T(\mathrm{t}) = T_a - Ce^{-Kt}$

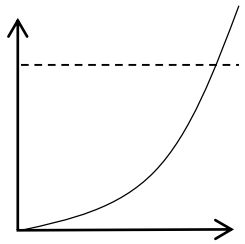
Zadanie

- T_a = 20 C
- T(0) = 80 C
- T(2) = 60 C

- Ile minut minie nim ochłodzi się herbata do 40 stopni?

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$



Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' =$

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha N'$

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

Gdy $K(C) = kC$

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

Gdy $K(C) = kC$

- $C' = -\alpha N'$

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

Gdy $K(C) = kC$

- $C' = -\alpha N'$
- $C = -\alpha N + C_0$

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

Gdy $K(C) = kC$

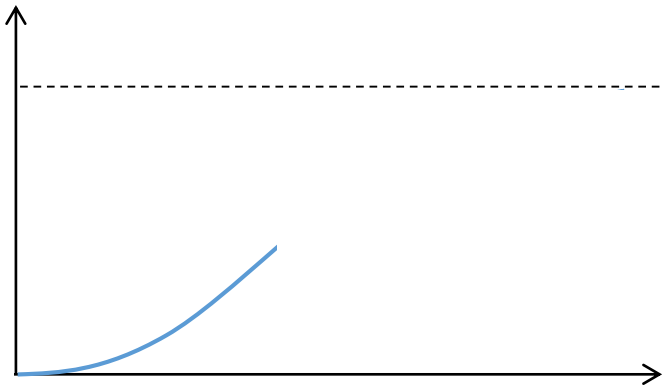
- $C' = -\alpha N'$
- $C = -\alpha N + C_0$
- $N' = k(C_0 - \alpha N)N$

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

Gdy $K(C) = kC$

- $C' = -\alpha N'$
- $C = -\alpha N + C_0$
- $N' = k(C_0 - \alpha N)N$

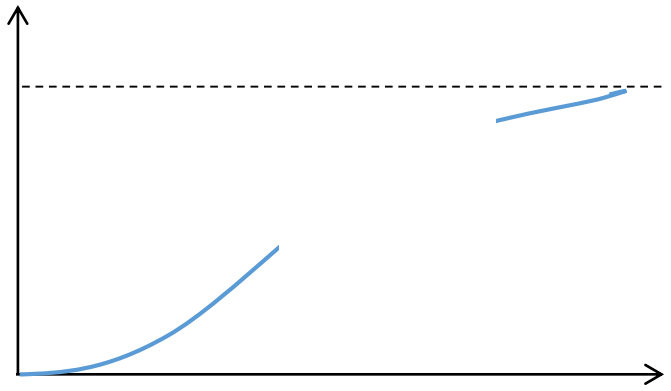


Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

Gdy $K(C) = kC$

- $C' = -\alpha N'$
- $C = -\alpha N + C_0$
- $N' = k(C_0 - \alpha N)N$



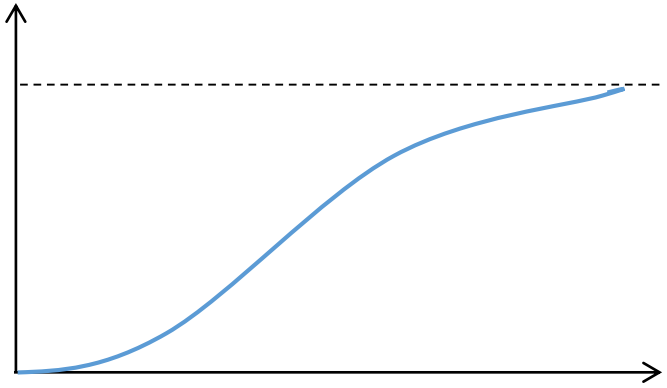
Dla N ~ C_0/α

Wzrost populacji N zależne od pokarmu C

- $N' = K(C)N$
- $C' = -\alpha K(C)N$

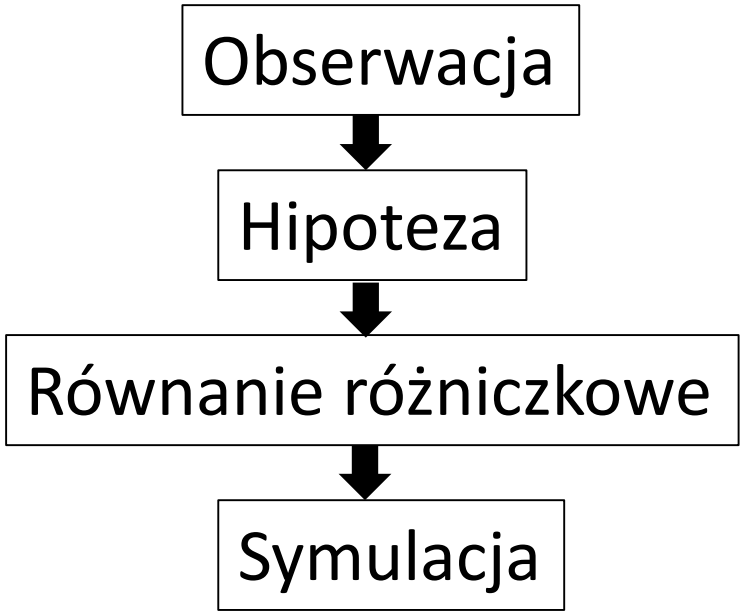
Gdy $K(C) = kC$

$$N \quad (t) = \frac{N_0 B}{N_0 + (B - N_0)e^{-rt}}$$

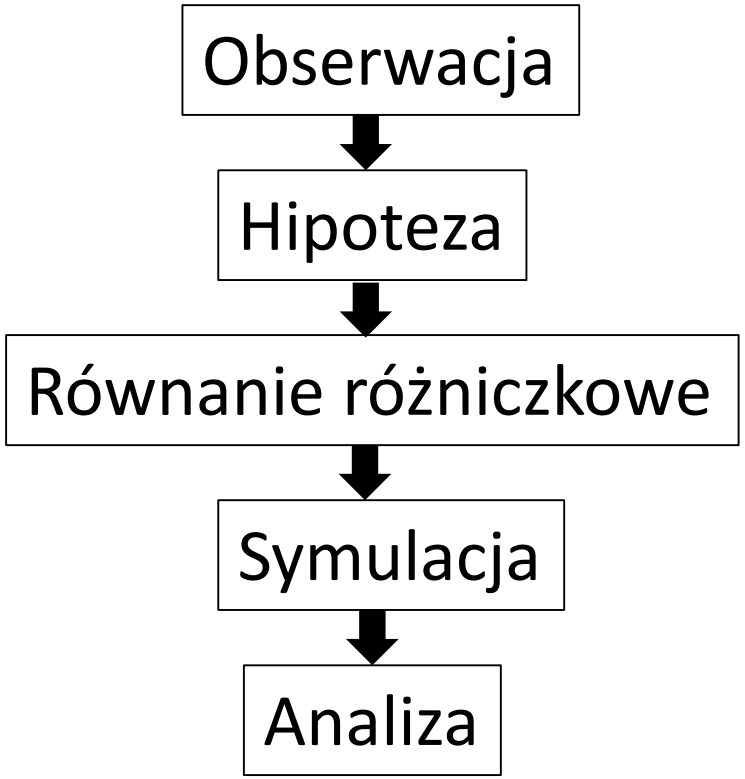


$$N_0 = N(0), B = \frac{C_0}{\alpha}, r = kC_0$$

Proces modelowania

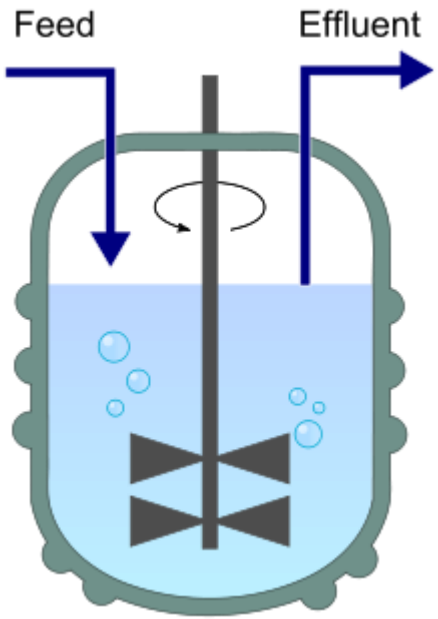


Proces modelowania



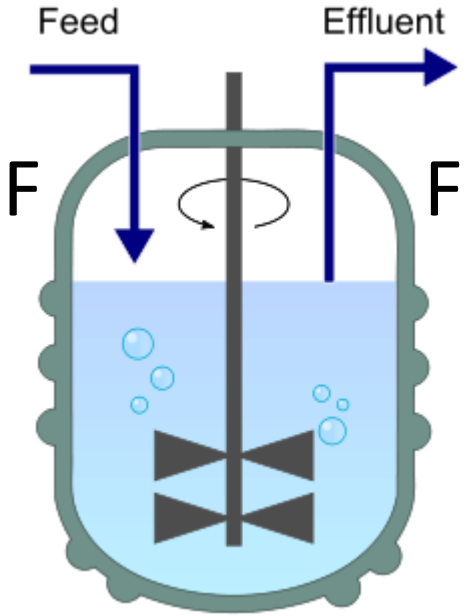
Chemostat

- $N(t)$: $koncentracja\ bakterii\ \frac{\#}{V}$
- $C(t)$: $koncentracja\ pokarmu\ \frac{masa}{V}$



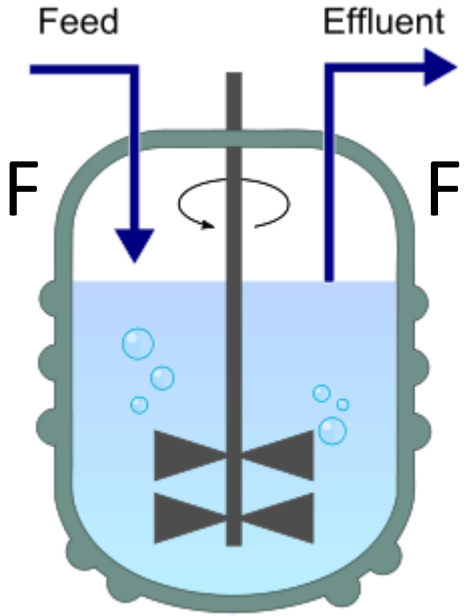
Chemostat model

- $N' = K(C)N$
- $C' = -\alpha K(C)N$



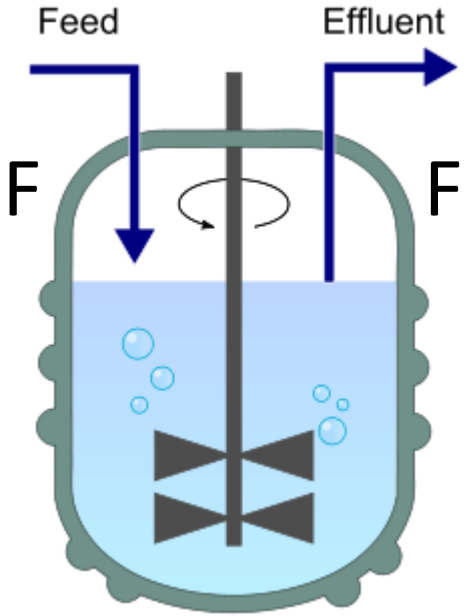
Chemostat model

- $N' = K(C)N - FN$
- $C' = -\alpha K(C)N$



Chemostat model

- $N' = K(C)N - FN$
- $C' = -\alpha K(C)N - FC + FC_0$



Analiza wymiarowa

$$N' = K(C)N - FN$$

Analiza wymiarowa

$$\frac{\#}{V \cdot t} \rightarrow N' = K(C)N - FN$$

Analiza wymiarowa

$$\frac{\#}{V \cdot t} \rightarrow N' = \overset{\frac{1}{t}}{K(C)}N - FN$$

Analiza wymiarowa

$$\frac{\#}{V \cdot t} \rightarrow N' = K(C)N - FN$$

$K(C)$: $\frac{\#}{V}$, $\frac{1}{t}$

Analiza wymiarowa

$$\frac{\#}{V \cdot t} \rightarrow N' = K(C)N - FN$$

$$K(C): \frac{\#}{V} \quad F: \frac{V}{t} \quad K(C): \frac{1}{t}$$

Analiza wymiarowa

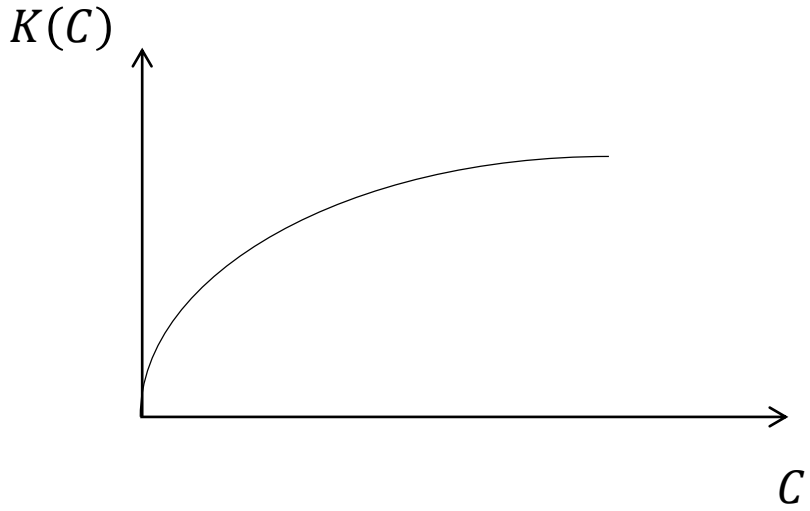
$$\frac{\#}{V \cdot t} \rightarrow N' = \underset{\frac{1}{t}}{K(C)}N - FN \leftarrow \frac{\#}{V}$$

$\frac{\#}{V}$ $\quad$ $\frac{V}{t}$

Poprawiony model

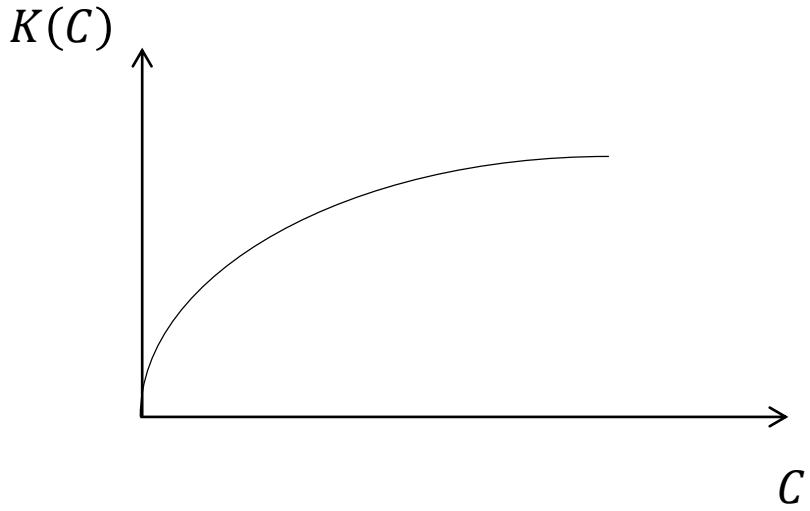
- $N' = K(C)N - \frac{F}{V}N$
- $C' = -\alpha K(C)N - \frac{F}{V}C + \frac{F}{V}C_0$

Funkcja pokarmu?



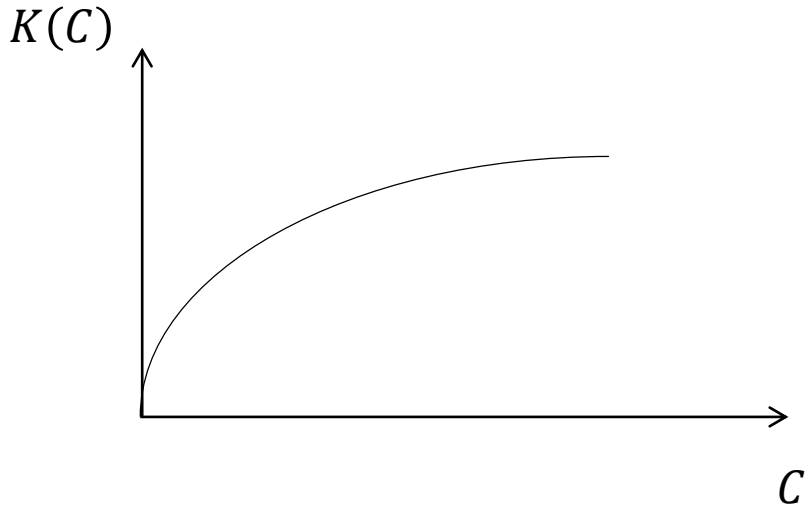
Funkcja pokarmu?

$$\frac{x}{1+x}$$



Funkcja Michaelis-Menten

$$K(C) = K_{max}\frac{C}{K_n + C}$$



Bardziej dokładny model

- $N' = K_{max} \frac{C}{K_n + C} N - \frac{F}{V} N$
- $C' = -\alpha K_{max} \frac{C}{K_n + C} N - \frac{F}{V} C + \frac{F}{V} C_0$

Nondimensionalization

- Pomysł:
  - pozbyć się jednostek parametrów
  - Zmniejszyć liczbę parametrów

Nondimensionalization

- Pomysl: $N = N^* \cdot \widehat{N}$

Nondimensionalization

- Pomysl:

$$N = N^* \cdot \widehat{N}$$

stara zmienna

nowa zmienna

stała

Nondimensionalization

- Pomysl: $N = N^* \cdot \widehat{N}$

stara zmienna

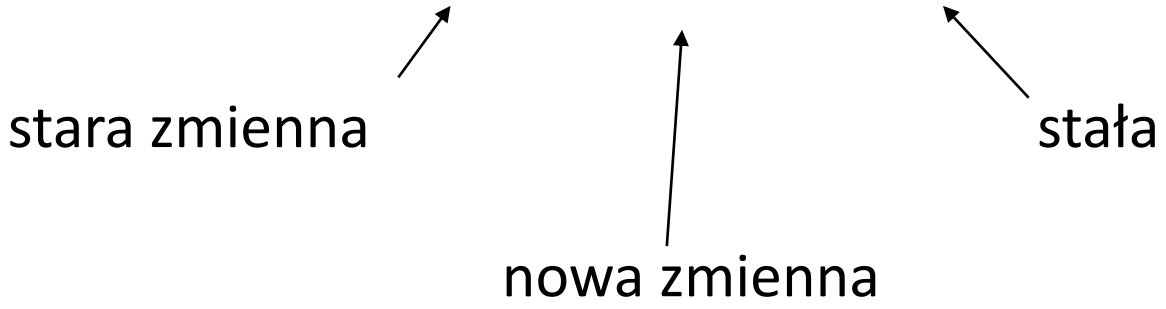
nowa zmienna

stała

- $\widehat{N}$ jest tak dobrany żeby miał taka jednostkę ($\frac{\#}{V}$) jak N, tzn. N^* nie ma jednostki

Nondimensionalization

- Pomysl: $N = N^* \cdot \widehat{N}$



- $\widehat{N}$ jest tak dobrany żeby miał taka jednostkę ($\frac{\#}{V}$) jak N, tzn. N^* nie ma jednostki
- Analogicznie dla C i t
- $C = C^* \cdot \hat{C}$
- $t = t^* \cdot \tau$

Zamiana zmiennych

- $N' = K_{max} \frac{C}{K_n + C} N - \frac{F}{V} N$
- $C' = -\alpha K_{max} \frac{C}{K_n + C} N - \frac{F}{V} C + \frac{F}{V} C_0$

Zamiana zmiennych

- $N' = K_{max} \frac{C}{K_n + C} N - \frac{F}{V} N$
- $C' = -\alpha K_{max} \frac{C}{K_n + C} N - \frac{F}{V} C + \frac{F}{V} C_0$

- $\frac{d(N^* \hat{N})}{d(t^* \tau)} = K_{max} \frac{C^* \hat{C}}{K_n + C^* \hat{C}} N^* \hat{N} - \frac{F}{V} N^* \hat{N}$
- $\frac{d(C^* \hat{C})}{d(t^* \tau)} = -\alpha K_{max} \frac{C^* \hat{C}}{K_n + C^* \hat{C}} N^* \hat{N} - \frac{F}{V} C^* \hat{C} + \frac{F}{V} C_0$

Upraszczanie

- $\frac{d(N^*\widehat{N})}{d(t^*\tau)} = K_{max}\frac{C^*\hat{C}}{K_n+C^*\hat{C}}N^*\widehat{N} - \frac{F}{V}N^*\widehat{N}$
- $\frac{d(C^*\hat{C})}{d(t^*\tau)} = -\alpha K_{max}\frac{C^*\hat{C}}{K_n+C^*\hat{C}}N^*\widehat{N} - \frac{F}{V}C^*\hat{C} + \frac{F}{V}C_0$

- $\frac{d(N^*)}{d(t^*)} = K_{max}\tau\frac{C^*}{\frac{K_n}{\widehat{C}}+C^*}N^* - \frac{F}{V}\tau N^*$
- $\frac{d(C^*)}{d(t^*)} = \frac{-\alpha K_{max}\tau\widehat{N}}{\hat{C}}\frac{C^*}{\frac{K_n}{\widehat{C}}+C^*}N^* - \frac{F}{V}\tau C^* + \frac{F}{V}\frac{\tau}{\hat{C}}C_0$

Wybór wartości dla $\hat{C}$, τ i $\widehat{N}$

- $\frac{d(N^*)}{d(t^*)} = K_{max}\tau \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau N^*$
- $\frac{d(C^*)}{d(t^*)} = \frac{-\alpha K_{max}\tau \widehat{N}}{\hat{C}} \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau C^* + \frac{F}{V}\frac{\tau}{\hat{C}} C_0$

Wybór wartości dla $\hat{C}$, τ i $\widehat{N}$

- $\frac{d(N^*)}{d(t^*)} = K_{max}\tau \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau N^*$
- $\frac{d(C^*)}{d(t^*)} = \frac{-\alpha K_{max}\tau \widehat{N}}{\hat{C}} \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau C^* + \frac{F}{V}\frac{\tau}{\hat{C}} C_0$

- Wybierz $\frac{K_n}{\hat{C}} = 1$

Wybór wartości dla $\hat{C}$, τ i $\hat{N}$

- $\frac{d(N^*)}{d(t^*)} = K_{max}\tau \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau N^*$
- $\frac{d(C^*)}{d(t^*)} = \frac{-\alpha K_{max}\tau \hat{N}}{\hat{C}} \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau C^* + \frac{F}{V}\frac{\tau}{\hat{C}} C_0$

- Wybierz $\frac{K_n}{\hat{C}} = 1$
- Wybierz $\frac{F}{V}\tau = 1$

Wybór wartości dla $\hat{C}$, τ i $\widehat{N}$

- $\frac{d(N^*)}{d(t^*)} = K_{max}\tau \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau N^*$
- $\frac{d(C^*)}{d(t^*)} = \frac{-\alpha K_{max}\tau \widehat{N}}{\hat{C}} \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau C^* + \frac{F}{V}\frac{\tau}{\hat{C}} C_0$

- Wybierz $\frac{K_n}{\hat{C}} = 1$
- Wybierz $\frac{F}{V}\tau = 1$
- Wybierz $\widehat{N} = \frac{\hat{C}}{\alpha K_{max}\tau}$

Wybór wartości dla $\hat{C}$, τ i $\widehat{N}$

- $\frac{d(N^*)}{d(t^*)} = K_{max}\tau \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau N^*$
- $\frac{d(C^*)}{d(t^*)} = \frac{-\alpha K_{max}\tau \widehat{N}}{\hat{C}} \frac{C^*}{\frac{K_n}{\hat{C}}+C^*} N^* - \frac{F}{V}\tau C^* + \frac{F}{V}\frac{\tau}{\hat{C}} C_0$

- Wybierz $\frac{K_n}{\hat{C}} = 1$
- Wybierz $\frac{F}{V}\tau = 1$
- Wybierz $\widehat{N} = \frac{\hat{C}}{\alpha K_{max}\tau} = \frac{K_n F}{\alpha K_{max} V}$

Nowy system

- $\frac{d(N^*)}{d(t^*)} = K_{max} \frac{V}{F} \frac{C^*}{1+C^*} N^* - N^*$
- $\frac{d(C^*)}{d(t^*)} = -\frac{C^*}{1+C^*} N^* - C^* + \frac{C_0}{K_n}$

Nowy system

- $\frac{d(N^*)}{d(t^*)} = K_{max}\frac{V}{F}\frac{C^*}{1+C^*}N^* - N^*$
- $\frac{d(C^*)}{d(t^*)} = -\frac{C^*}{1+C^*}N^* - C^* + \frac{C_0}{K_n}$

- $\alpha_1 = K_{max}\frac{V}{F}$

Nowy system

- $\frac{d(N^*)}{d(t^*)} = K_{max}\frac{V}{F}\frac{C^*}{1+C^*}N^* - N^*$
- $\frac{d(C^*)}{d(t^*)} = -\frac{C^*}{1+C^*}N^* - C^* + \frac{C_0}{K_n}$

- $\alpha_1 = K_{max}\frac{V}{F}$
- $\alpha_2 = \frac{C_0}{K_n}$

Nowy system

- $\frac{d(N^*)}{d(t^*)} = \alpha_1 \frac{C^*}{1+C^*} N^* - N^*$
- $\frac{d(C^*)}{d(t^*)} = -\frac{C^*}{1+C^*} N^* - C^* + \alpha_2$

- $\alpha_1 = K_{max} \frac{V}{F}$
- $\alpha_2 = \frac{C_0}{K_n}$

Stany stacjonarne i homeostaza

- $\frac{d(N)}{d(t)} = 0, \quad \frac{d(C)}{d(t)} = 0$

$$\frac{d(N)}{d(t)} = \alpha_1 \frac{C}{1+C} N - N$$

$$\frac{d(C)}{d(t)} = -\frac{C}{1+C} N - C + \alpha_2$$

Stany stacjonarne i homeostaza

- $\frac{d(N)}{d(t)} = 0, \quad \frac{d(C)}{d(t)} = 0$

- $\alpha_1 \frac{C}{1+C} N - N = 0$

- $-\frac{C}{1+C} N - C + \alpha_2 = 0$

$$\frac{d(N)}{d(t)} = \alpha_1 \frac{C}{1+C} N - N$$

$$\frac{d(C)}{d(t)} = -\frac{C}{1+C} N - C + \alpha_2$$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

- Przypadek 1: $\bar{N} = 0$
- Wtedy $\bar{C} = \alpha_2$
- Czyli $(\bar{N}, \bar{C})$=$(0, \alpha_2)$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

- Przypadek 1: $\bar{N} = 0$
- Wtedy $\bar{C} = \alpha_2$
- Czyli $(\bar{N}, \bar{C})=(0, \alpha_2)$

- Przypadek 2: $\bar{N} > 0$
- $\alpha_1 \frac{\bar{C}}{1+\bar{C}} = 1$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

- Przypadek 1: $\bar{N} = 0$
- Wtedy $\bar{C} = \alpha_2$
- Czyli $(\bar{N}, \bar{C})=(0, \alpha_2)$

- Przypadek 2: $\bar{N} > 0$
- $\alpha_1 \frac{\bar{C}}{1+\bar{C}} = 1$
- $\alpha_1 \bar{C} = 1 + \bar{C}$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

- Przypadek 1: $\bar{N} = 0$
- Wtedy $\bar{C} = \alpha_2$
- Czyli $(\bar{N}, \bar{C})=(0, \alpha_2)$

- Przypadek 2: $\bar{N} > 0$
- $\alpha_1 \frac{\bar{C}}{1+\bar{C}} = 1$
- $\alpha_1 \bar{C} = 1 + \bar{C}$
- $\bar{C} = \frac{1}{\alpha_1 - 1}$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

- $-\frac{1}{\alpha_1} \bar{N} - \frac{1}{\alpha_1 - 1} + \alpha_2 = 0$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

- $-\frac{1}{\alpha_1} \bar{N} - \frac{1}{\alpha_1 - 1} + \alpha_2 = 0$

- $\bar{N} = \alpha_1 (\alpha_2 - \frac{1}{\alpha_1 - 1})$

$$\alpha_1 \frac{C}{1+C} N - N = 0$$

$$-\frac{C}{1+C} N - C + \alpha_2 = 0$$

- $-\frac{1}{\alpha_1} \bar{N} - \frac{1}{\alpha_1 - 1} + \alpha_2 = 0$
- $\bar{N} = \alpha_1 (\alpha_2 - \frac{1}{\alpha_1 - 1})$
- $(\bar{N}, \bar{C}) = (\alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1})$

Warunki dla stanów stacjonarnych

- $\frac{d(N)}{d(t)} = \alpha_1 \frac{C}{1+C} N - N$
- $\frac{d(C)}{d(t)} = -\frac{C}{1+C} N - C + \alpha_2$

- $(\bar{N}, \bar{C}) = (\alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1})$

Warunki dla stanów stacjonarnych

- $\frac{d(N)}{d(t)} = \alpha_1 \frac{C}{1+C} N - N$
- $\frac{d(C)}{d(t)} = -\frac{C}{1+C} N - C + \alpha_2$

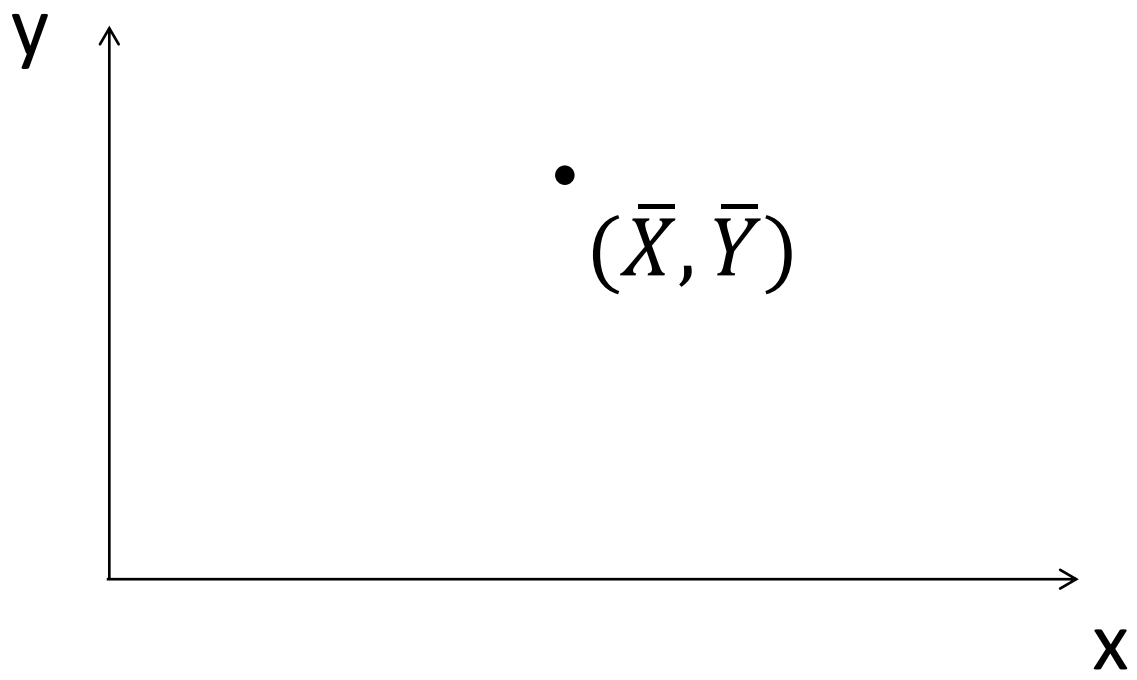
- $(\bar{N}, \bar{C}) = (\alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1})$
- $\alpha_1 > 1$

Warunki dla stanów stacjonarnych

- $\frac{d(N)}{d(t)} = \alpha_1 \frac{C}{1+C} N - N$
- $\frac{d(C)}{d(t)} = -\frac{C}{1+C} N - C + \alpha_2$

- $(\bar{N}, \bar{C}) = (\alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1})$
- $\alpha_1 > 1$
- $\alpha_2 > \frac{1}{\alpha_1 - 1}$

Co się staje z naszym rozwiązaniem gdy zaczynamy blisko stanu stacjonarnego?



Analiza stabilności (aproksymacja liniowa)

- $\frac{d(N)}{d(t)} = \alpha_1 \frac{C}{1+C} N - N = F(N, C)$
- $\frac{d(C)}{d(t)} = -\frac{C}{1+C} N - C + \alpha_2 = G(N, C)$

Analiza stabilności (aproksymacja liniowa)

- $\frac{d(N)}{d(t)} = \alpha_1 \frac{C}{1+C} N - N = F(N, C)$
- $\frac{d(C)}{d(t)} = -\frac{C}{1+C} N - C + \alpha_2 = G(N, C)$

- Macierz Jacobiego
- $J = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial C} \end{pmatrix}$

$y' = Jy$ (J w punkcie $(\bar{N}, \bar{C})$)

Analiza stabilności (aproksymacja liniowa)

- $\frac{d(N)}{d(t)} = \alpha_1 \frac{C}{1+C} N - N = F(N, C)$
- $\frac{d(C)}{d(t)} = -\frac{C}{1+C} N - C + \alpha_2 = G(N, C)$

- Macierz Jacobiego
- $J = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial C} \end{pmatrix}$

$y' = Jy$ (J w punkcie $(\bar{N}, \bar{C})$)

Rozwiązanie dla y(t) znajdziemy za pomocy **wartości** i **wektorów własnych** λ_i, v_i, np. $y(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

Specjalny przypadek w 2D

- $y' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} y$

- $\tau = a + d$
- $D = \det(A) = ad - cb$

- $Re(\lambda_1) < 0, \quad Re(\lambda_2) < 0 \quad \leftrightarrow \quad \tau < 0, \quad D > 0$

Analiza stabilności (aproksymacja liniowa)

- $J = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial C} \end{pmatrix}$

Analiza stabilności (aproksymacja liniowa)

- $J = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial C} \end{pmatrix} = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$

Analiza stabilności (aproksymacja liniowa)

- $J = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial C} \end{pmatrix} = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$

- W punkcie $(\bar{N}, \bar{C})$
- $\frac{\partial F}{\partial N} = \alpha_1 \frac{\bar{C}}{1+\bar{C}} - 1$

Analiza stabilności (aproksymacja liniowa)

- $J = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial C} \end{pmatrix} = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$

- W punkcie $(\bar{N}, \bar{C})$
- $\frac{\partial F}{\partial N} = \alpha_1 \frac{\bar{C}}{1+\bar{C}} - 1 = \alpha_1 \frac{\frac{1}{\alpha_1 - 1}}{1+\frac{1}{\alpha_1 - 1}} - 1$

Analiza stabilności (aproksymacja liniowa)

- $J = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial C} \end{pmatrix} = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$

- W punkcie $(\bar{N}, \bar{C})$
- $\frac{\partial F}{\partial N} = \alpha_1 \frac{\bar{C}}{1+\bar{C}} - 1 = \alpha_1 \frac{\frac{1}{\alpha_1 - 1}}{1+\frac{1}{\alpha_1 - 1}} - 1 = \alpha_1 \frac{1}{(\alpha_1 - 1)+1} - 1$

Analiza stabilności (aproksymacja liniowa)

- $J = \begin{pmatrix} \frac{\partial F}{\partial N} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial N} & \frac{\partial G}{\partial C} \end{pmatrix} = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$

- W punkcie $(\bar{N}, \bar{C})$
- $\frac{\partial F}{\partial N} = \alpha_1 \frac{\bar{C}}{1+\bar{C}} - 1 = \alpha_1 \frac{\frac{1}{\alpha_1 - 1}}{1+\frac{1}{\alpha_1 - 1}} - 1 = \alpha_1 \frac{1}{(\alpha_1 - 1)+1} - 1 = 0$

$$J = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$$

- $\tau = 0 - \frac{N}{(1+C)^2} - 1$

$$J = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$$

- $\tau = 0 - \frac{N}{(1+C)^2} - 1 < 0$

$$J = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$$

- $\tau = 0 - \frac{N}{(1+C)^2} - 1 < 0$
- $D = 0 - \left(-\frac{\bar{C}}{1+\bar{C}}\right) \alpha_1 \frac{1}{(1+\bar{C})^2} \bar{N}$

$$J = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$$

- $\tau = 0 - \frac{N}{(1+C)^2} - 1 < 0$

- $D = 0 - \left(-\frac{\bar{C}}{1+\bar{C}}\right) \alpha_1 \frac{1}{(1+\bar{C})^2} \bar{N} = \frac{\bar{N}}{(1+\bar{C})^2}$

$$J = \begin{pmatrix} \alpha_1 \frac{C}{1+C} - 1 & \alpha_1 \frac{1}{(1+C)^2} N \\ -\frac{C}{1+C} & -\frac{N}{(1+C)^2} - 1 \end{pmatrix}$$

- $\tau = 0 - \frac{N}{(1+C)^2} - 1 < 0$
- $D = 0 - \left(-\frac{\bar{C}}{1+\bar{C}}\right) \alpha_1 \frac{1}{(1+\bar{C})^2} \bar{N} = \frac{\bar{N}}{(1+\bar{C})^2} > 0$
- Czyli stan stacjonarny jest stabilny!

Co oznaczają nasze warunki?

- $\alpha_1 > 1 \quad \leftrightarrow \quad \frac{1}{K_{max}} < \frac{V}{F}$

- $\alpha_1 = K_{max} \frac{V}{F}$

Co oznaczają nasze warunki?

- $\alpha_1 > 1 \quad \leftrightarrow \quad \frac{1}{K_{max}} < \frac{V}{F}$

Najszybszy czas rozmnażania

- $\alpha_1 = K_{max} \frac{V}{F}$

Co oznaczają nasze warunki?

- $\alpha_1 > 1 \quad \leftrightarrow \quad \frac{1}{K_{max}} < \frac{V}{F}$

Najszybszy czas rozmnażania

Czas żeby cały pojemnik wylać

- $\alpha_1 = K_{max} \frac{V}{F}$

Co oznaczają nasze warunki?

- $\alpha_2 > \frac{1}{\alpha_1 - 1} \quad \leftrightarrow \quad \frac{C_0}{K_n} > \bar{C}^*_1$

- $\alpha_2 = \frac{C_0}{K_n}$

Co oznaczają nasze warunki?

- $\alpha_2 > \frac{1}{\alpha_1 - 1} \quad \leftrightarrow \quad \frac{C_0}{K_n} > \bar{C}^*{}_1$

- $C_0 > \bar{C}^*{}_1 K_n$

- $\alpha_2 = \frac{C_0}{K_n}$

Co oznaczają nasze warunki?

- $\alpha_2 > \frac{1}{\alpha_1 - 1} \quad \leftrightarrow \quad \frac{C_0}{K_n} > \bar{C}^*{}_1$

- $C_0 > \bar{C}^*{}_1 K_n = \bar{C}^*{}_1 \hat{C}$

- $\alpha_2 = \frac{C_0}{K_n}$

Co oznaczają nasze warunki?

- $\alpha_2 > \frac{1}{\alpha_1 - 1} \quad \leftrightarrow \quad \frac{C_0}{K_n} > \bar{C}^*{}_1$

- $C_0 > \bar{C}^*{}_1 K_n = \bar{C}^*{}_1 \hat{C} = \bar{C}_1$

- $\alpha_2 = \frac{C_0}{K_n}$

Co oznaczają nasze warunki?

- $\alpha_2 > \frac{1}{\alpha_1 - 1} \quad \leftrightarrow \quad \frac{C_0}{K_n} > \bar{C}^*{}_1$

- $C_0 > \bar{C}^*{}_1 K_n = \bar{C}^*{}_1 \hat{C} = \bar{C}_1$

↑

Koncentracja wpływającego pokarmu

- $\alpha_2 = \frac{C_0}{K_n}$

Co oznaczają nasze warunki?

- $\alpha_2 > \frac{1}{\alpha_1 - 1} \quad \leftrightarrow \quad \frac{C_0}{K_n} > \bar{C}^*{}_1$

- $C_0 > \bar{C}^*{}_1 K_n = \bar{C}^*{}_1 \hat{C} = \bar{C}_1$

Koncentracja wpływającego pokarmu

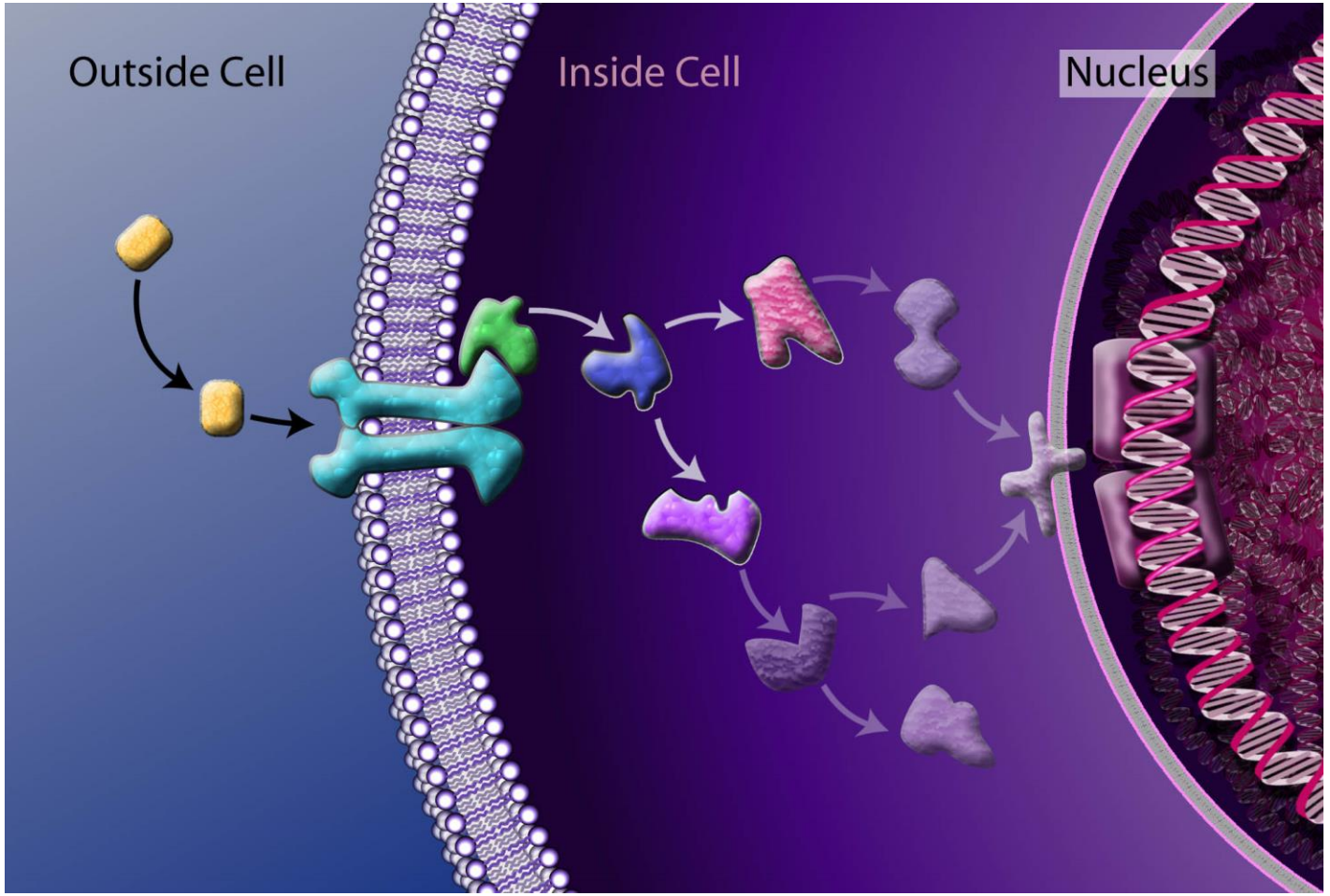
Koncentracja pokarmu w homeostazie

- $\alpha_2 = \frac{C_0}{K_n}$

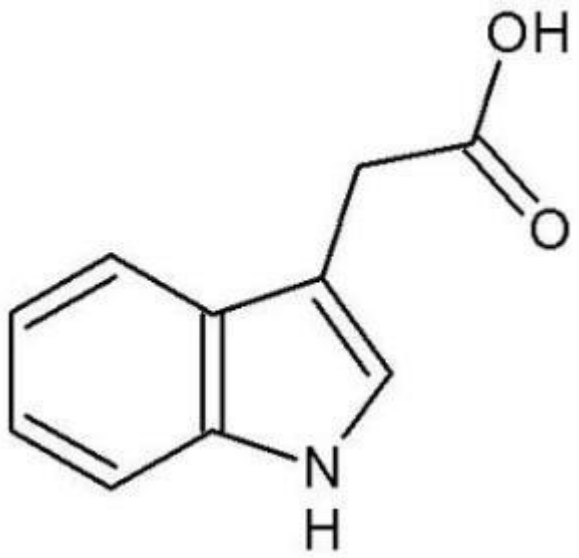
Proces analizy

- Analiza wymiarowa
- Nondimensionalization
- Stany stacjonarne
- Analiza stabilności
- Odniesienie się do rzeczywistości

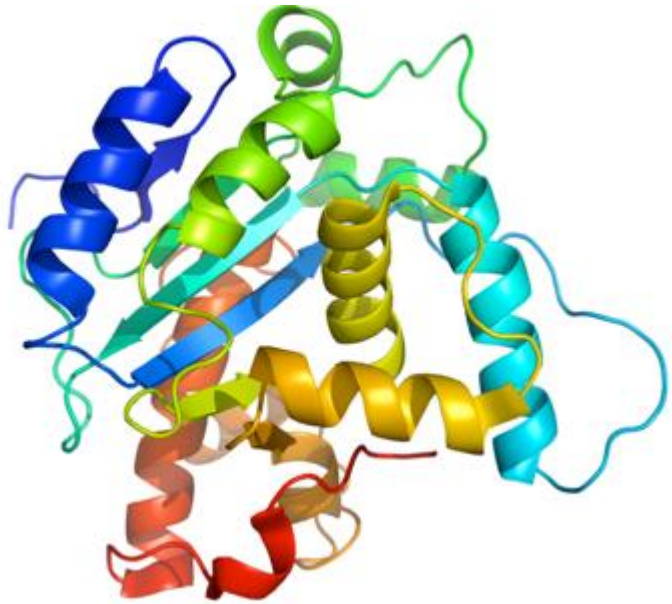
Przekaz informacji w systemach biologicznych



Sygnały

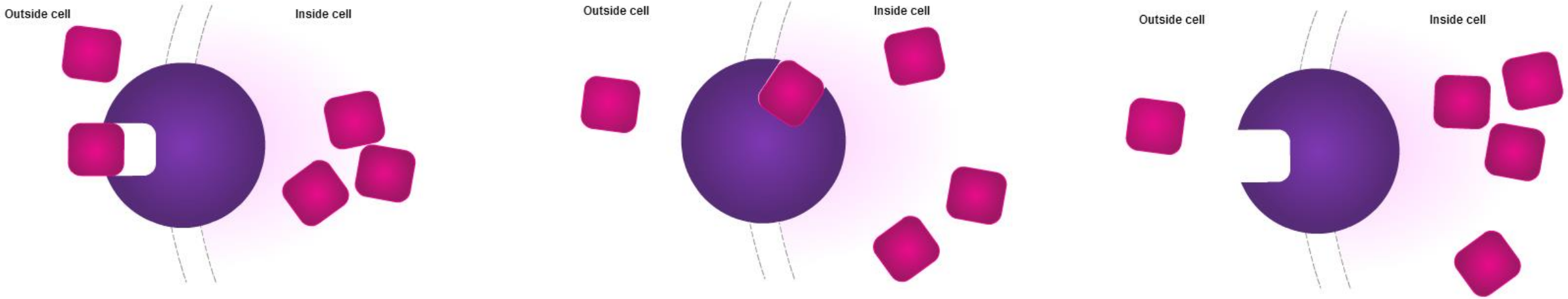


Hormony



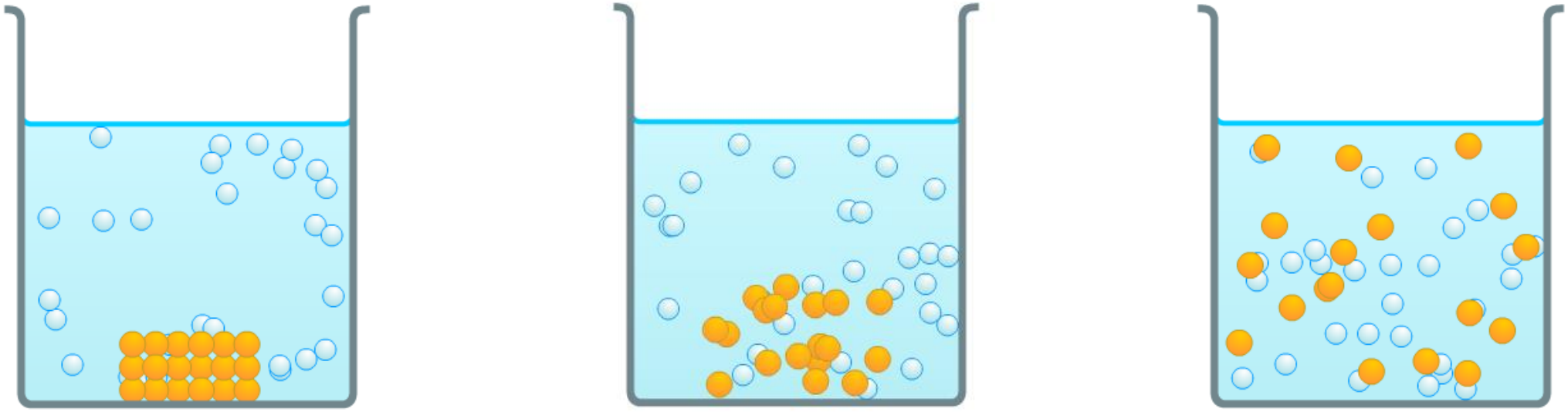
Białka

Aktywny/pasywny transport



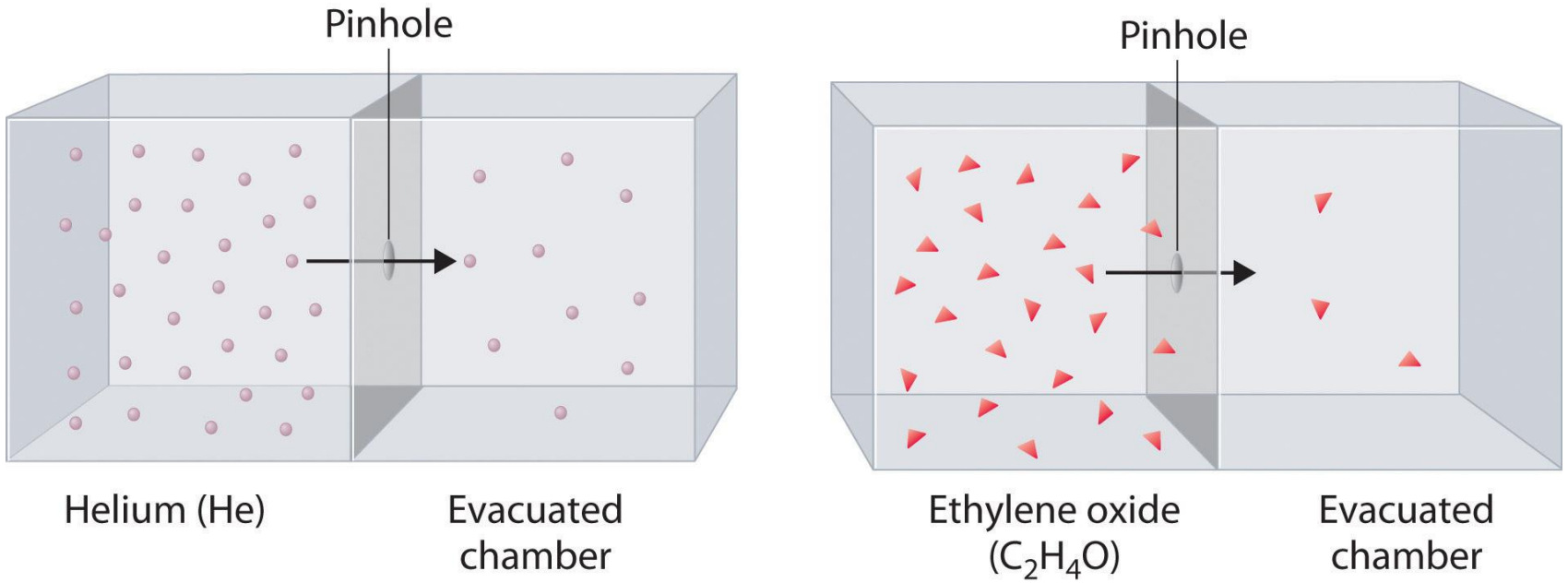
http://www.abpischools.org.uk/

Dyfuzja



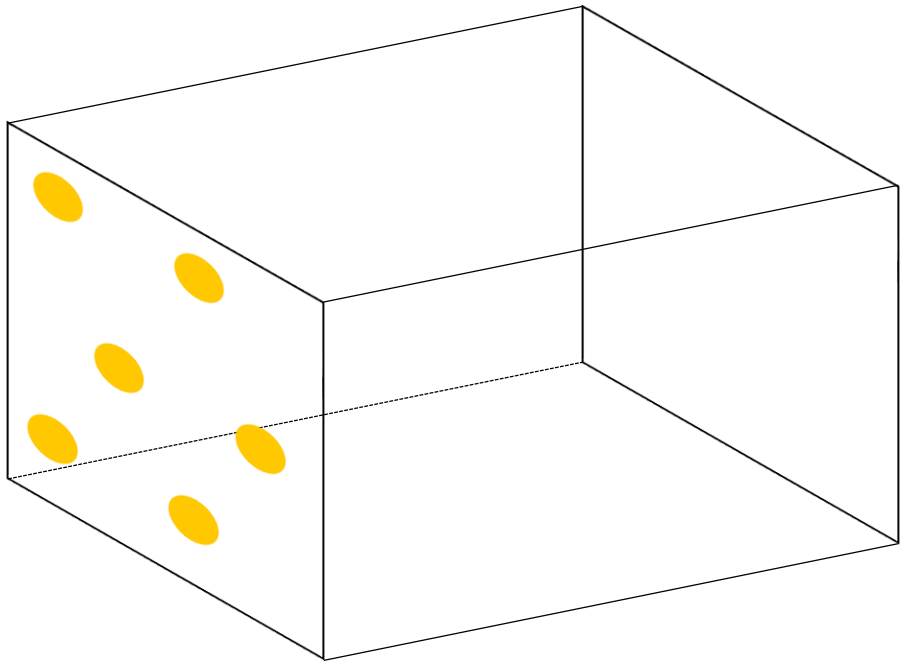
http://www.abpischools.org.uk/

Prawo Grahama

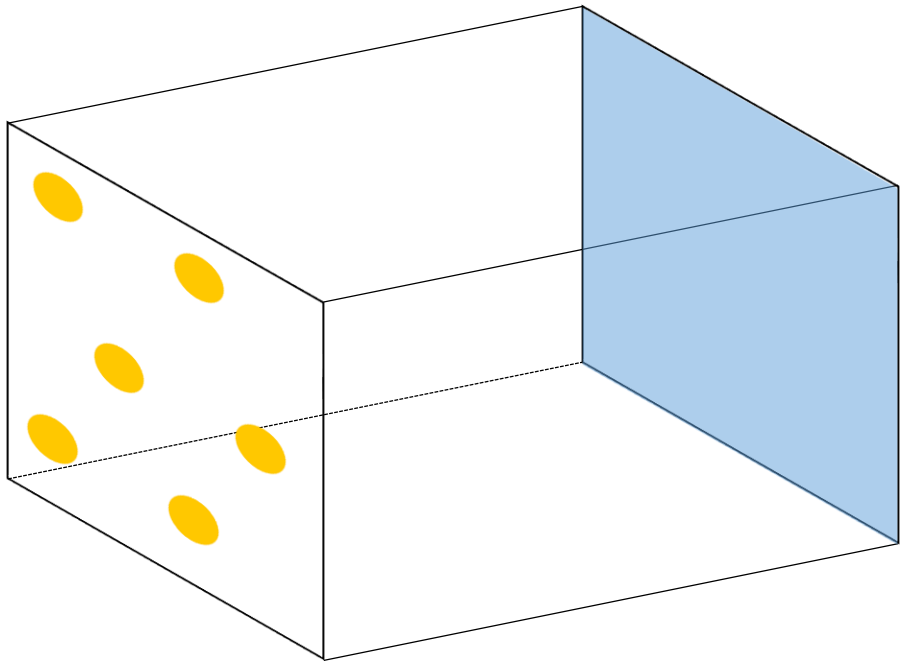


$$\frac{Rate_A}{Rate_B} = \sqrt{\frac{Molar\ Mass_B}{Molar\ Mass_A}}$$

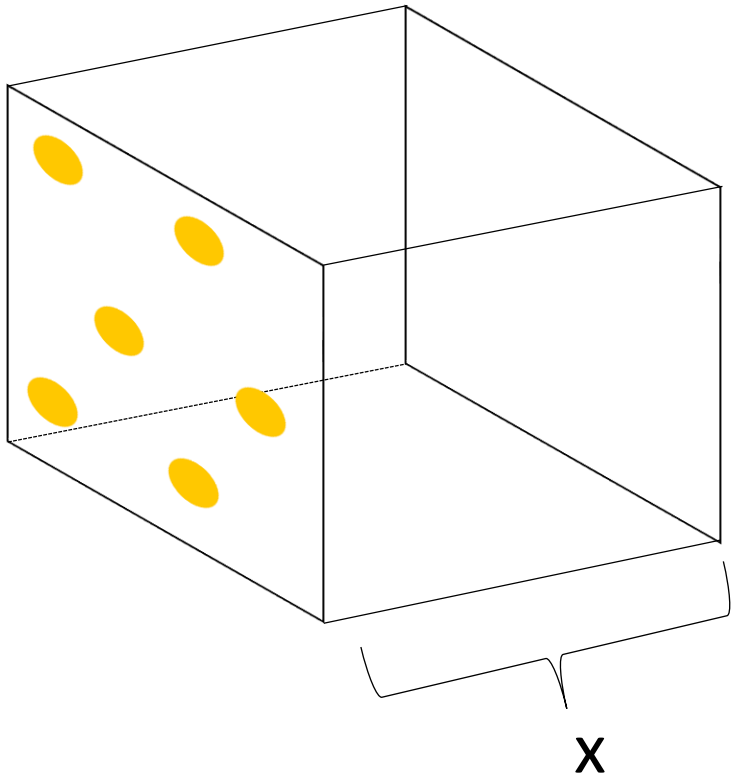
Dyfuzja



Dyfuzja

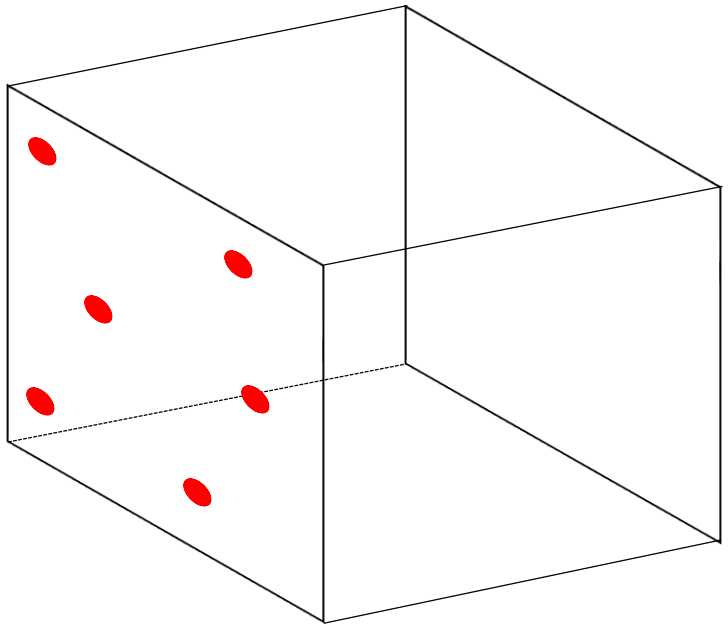


Dyfuzja



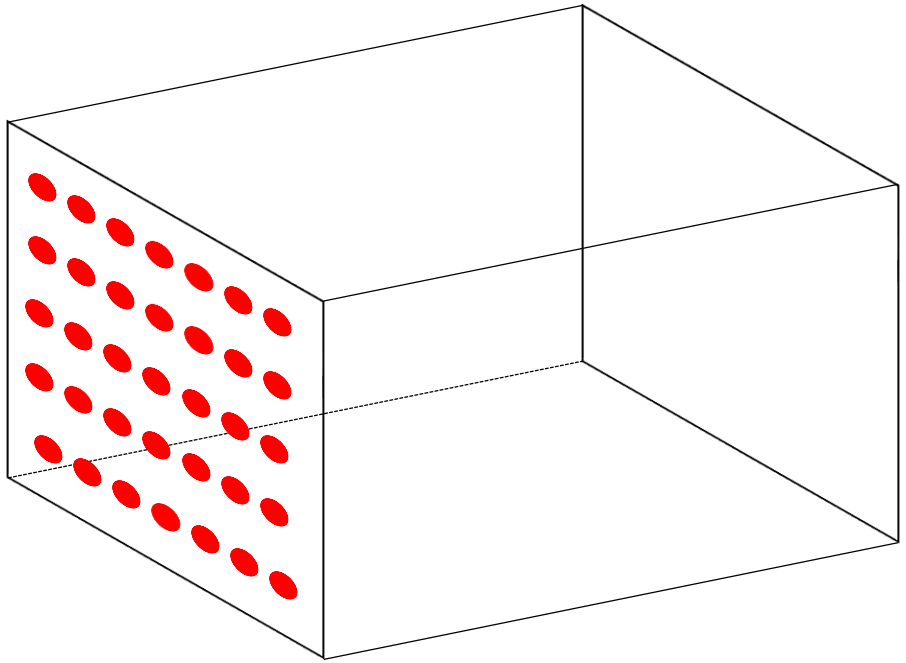
Odległość x

Dyfuzja



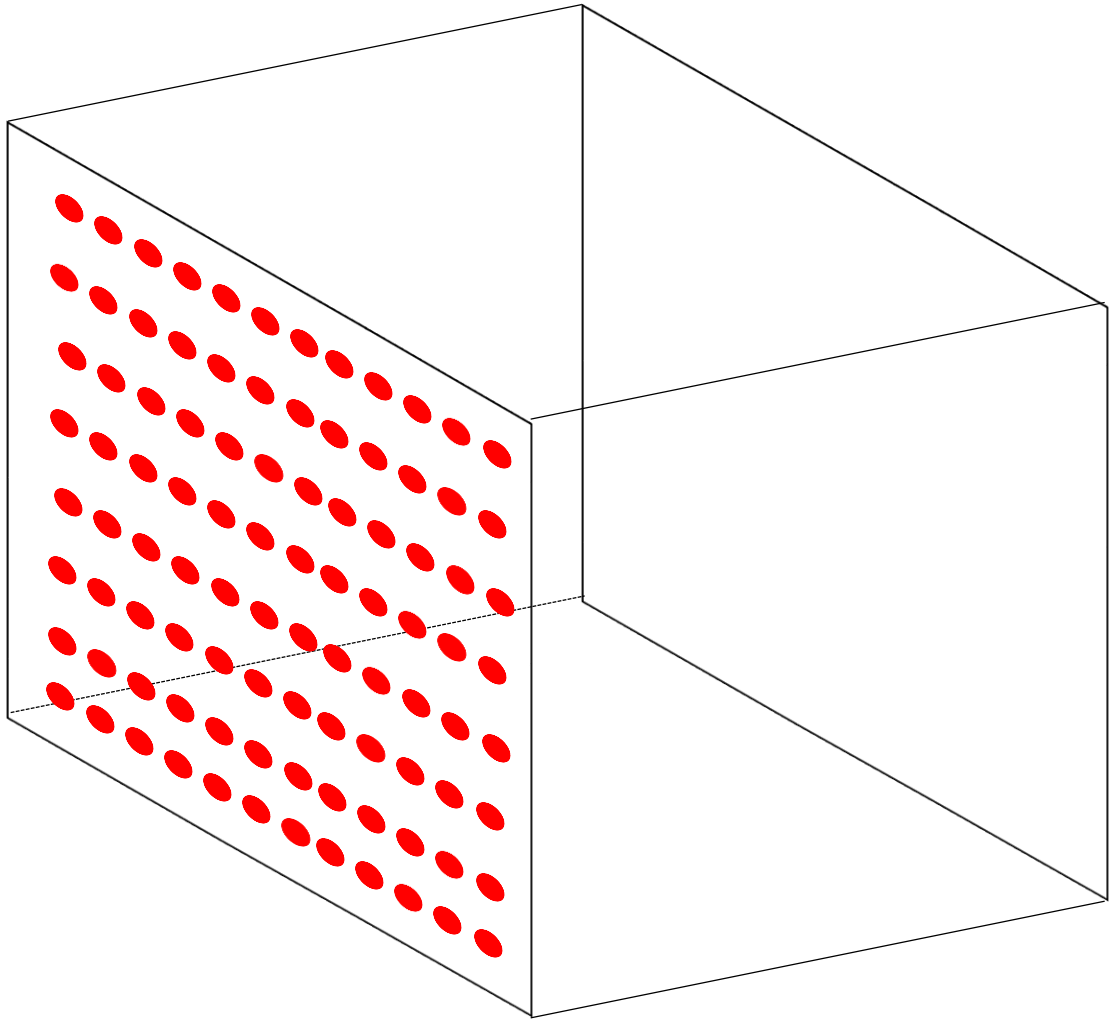
”Molarna masa” D
(współczynnik dyfuzji)

Dyfuzja



Gęstość (koncentracja) c

Dyfuzja



Powierzchnia A

Pierwsze prawo Ficka

$$\dot{V} = \frac{(c_1 - c_2) \cdot A \cdot D}{x}$$

Pierwsze prawo Ficka

$$\dot{V} = \frac{(c_1 - c_2) \cdot A \cdot D}{x}$$

$$\frac{\dot{V}}{A} = \frac{(c_1 - c_2)}{x} D$$

Pierwsze prawo Ficka

$$\dot{V} = \frac{(c_1 - c_2) \cdot A \cdot D}{x}$$

$$\frac{\dot{V}}{A} = \frac{(c_1 - c_2)}{x} D$$

Strumień pola

Gradient

Pierwsze prawo Ficka

$$\dot{V} = \frac{(c_1 - c_2) \cdot A \cdot D}{x}$$

$$\frac{\dot{V}}{A} = -D\nabla c$$

Strumień pola

Gradient

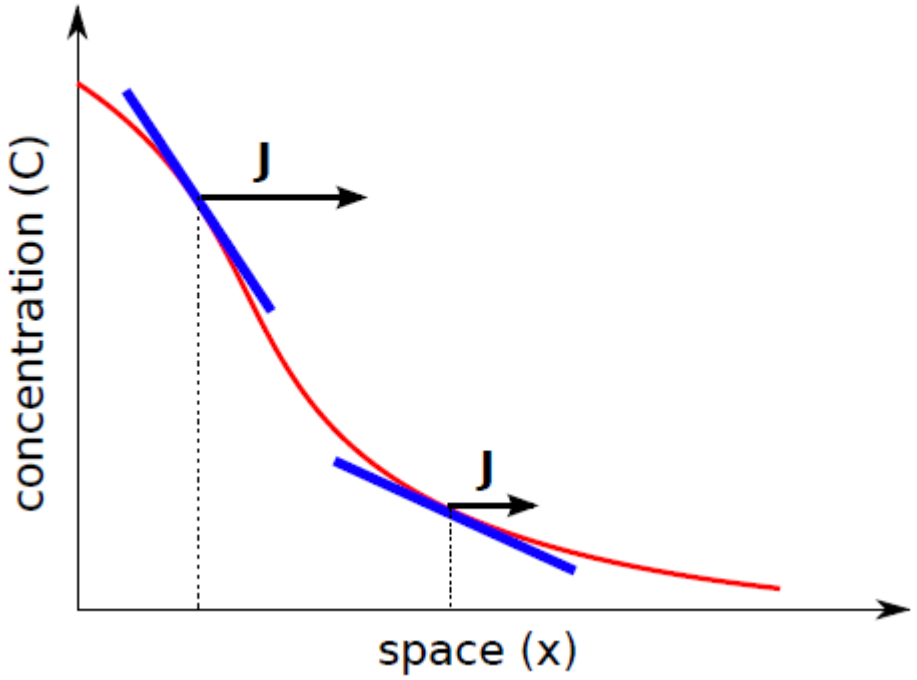
Pierwsze prawo Ficka

$$\dot{V} = \frac{(c_1 - c_2) \cdot A \cdot D}{x}$$

$$J = -D\nabla c$$

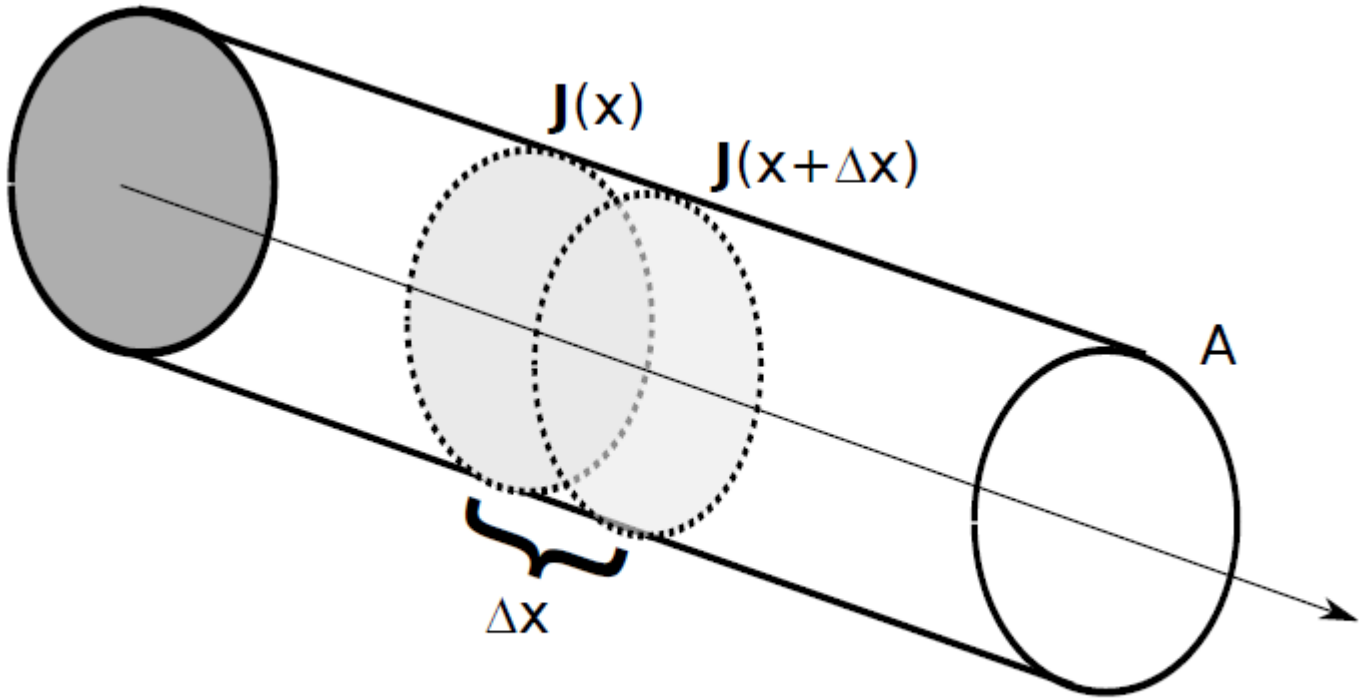
Strumień pola

Gradient



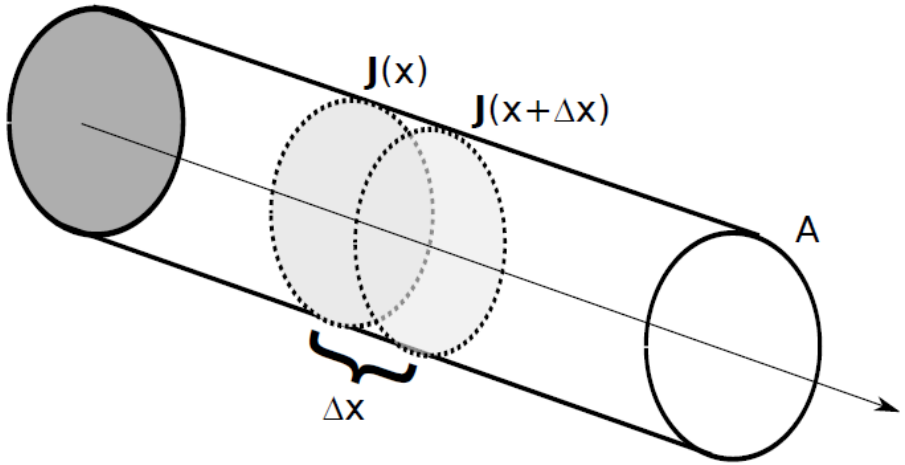
Równanie ciągłości

- Jak zmienia się liczba cząsteczek w danej pozycji?
- W przypadku cylindra od punktu x do $x + \Delta x$:



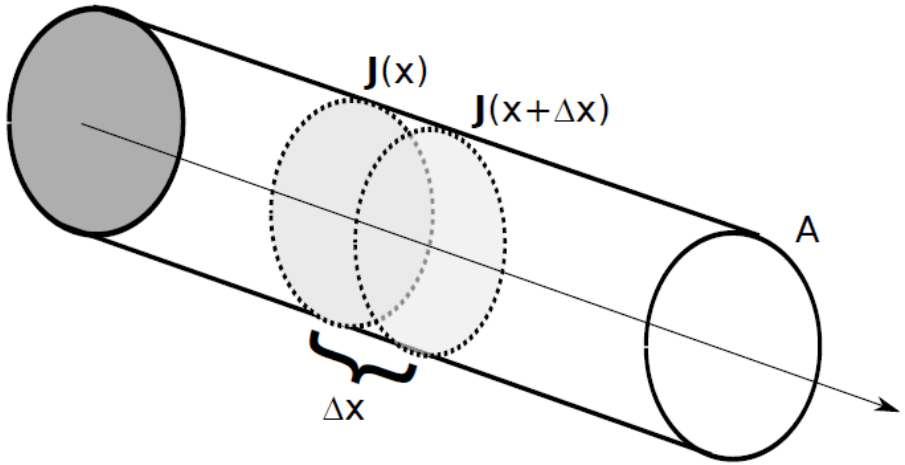
Równanie ciągłości

- Liczba cząsteczek to:
  - $N = cA\Delta x$
  - Gdzie A powierzchnia przekroju cylindra
  - c, koncentracja cząsteczek
  - Δx, długość cylindra



Równanie ciągłości

- Liczba cząsteczek to:
  - $N = cA\Delta x$
  - Gdzie A powierzchnia przekroju cylindra
  - c, koncentracja cząsteczek
  - Δx, długość cylindra
- Wtedy zmiana liczby cząsteczek w czasie t jest dane przez przepływy cząsteczek w pozycji x i $x + \Delta x$:
- $\frac{\partial(cA\Delta x)}{\partial t} = AJ(x) - AJ(x + \Delta x) \pm \sigma A\Delta x$



Równanie ciągłości

- $\frac{\partial(cA\Delta x)}{\partial t} = AJ(x) - AJ(x + \Delta x) \pm \sigma A\Delta x$

Równanie ciągłości

- $\frac{\partial(cA\Delta x)}{\partial t} = AJ(x) - AJ(x+\Delta x) \pm \sigma A\Delta x$

- $\frac{\partial c(x,t)}{\partial t} = \frac{J(x)-J(x+\Delta x)}{\Delta x} \pm \sigma$

Równanie ciągłości

- $\frac{\partial(cA\Delta x)}{\partial t} = AJ(x) - AJ(x+\Delta x) \pm \sigma A\Delta x$

- $\frac{\partial c(x,t)}{\partial t} = \frac{J(x)-J(x+\Delta x)}{\Delta x} \pm \sigma$

- Gdy Δx zbliża się do zera dostajemy:
- $\frac{\partial c}{\partial t} = \frac{\partial J}{\partial x} \pm \sigma$

Równanie ciągłości

- $\frac{\partial(cA\Delta x)}{\partial t} = AJ(x) - AJ(x+\Delta x) \pm \sigma A\Delta x$

- $\frac{\partial c(x,t)}{\partial t} = \frac{J(x)-J(x+\Delta x)}{\Delta x} \pm \sigma$

- Gdy Δx zbliża się do zera dostajemy:
- $\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x} \pm \sigma$
- W przypadku wielowymiarowym: $\frac{\partial c}{\partial t} = -\nabla \cdot \vec{J} \pm \sigma$

Drugie prawo Ficka

- $\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x}$

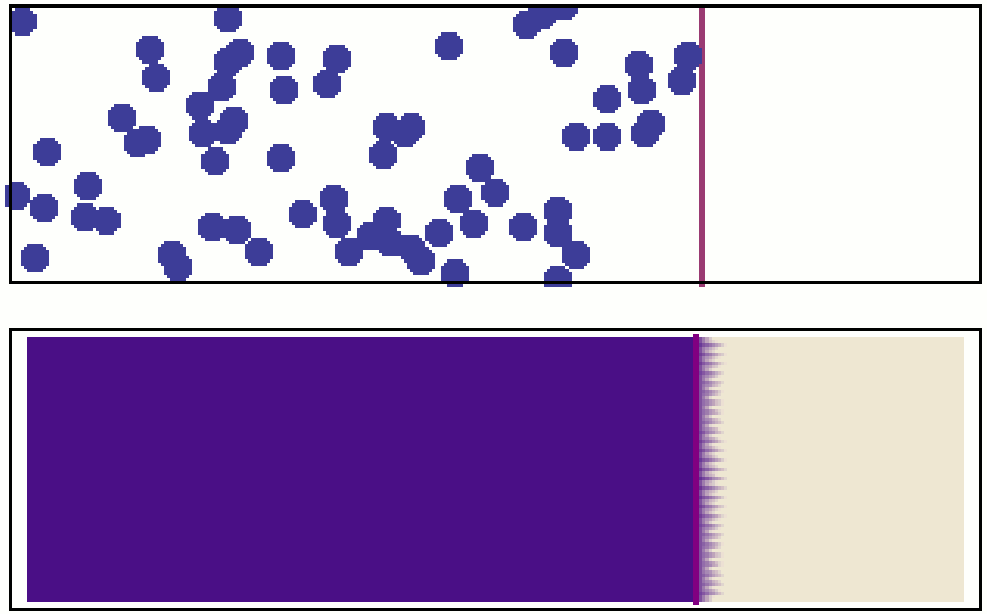
Drugie prawo Ficka

- $\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x} = \frac{\partial\left(D\frac{\partial c}{\partial x}\right)}{\partial x}$

Drugie prawo Ficka

- $\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x} = \frac{\partial\left(D\frac{\partial c}{\partial x}\right)}{\partial x} = D\,\frac{\partial^2 c}{\partial x^2}$

Drugie prawo Ficka: $\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$

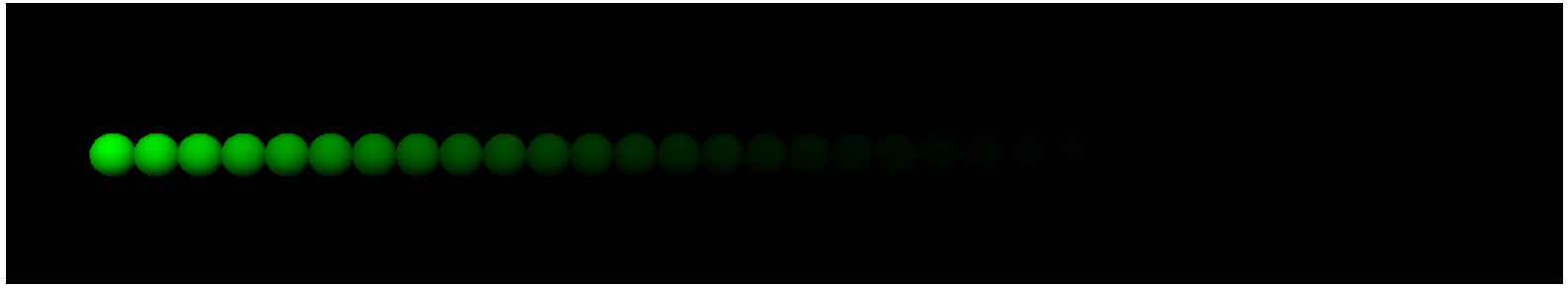


Dyskretny model

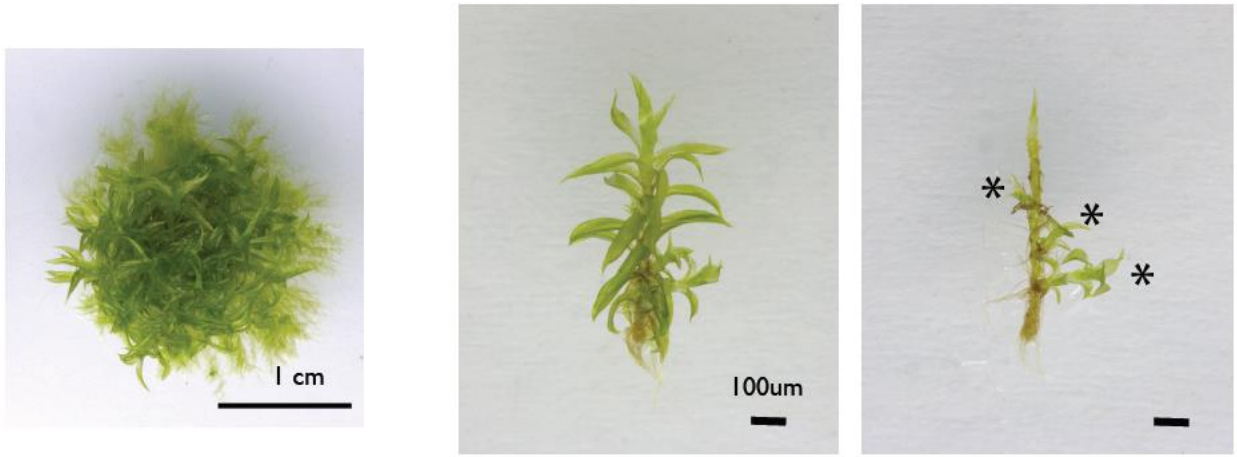
Ciągły model

Zadanie

- Stwórz szereg kul we vPython, tak żeby każda kula dotykała sąsiednie.
- Najbardziej lewa kula stało produkuje jakąś substancje. Wyraź koncentracje substancji intensywnością koloru kuli.
- Zaimplementuj dyfuzje substancji pomiędzy kulami.

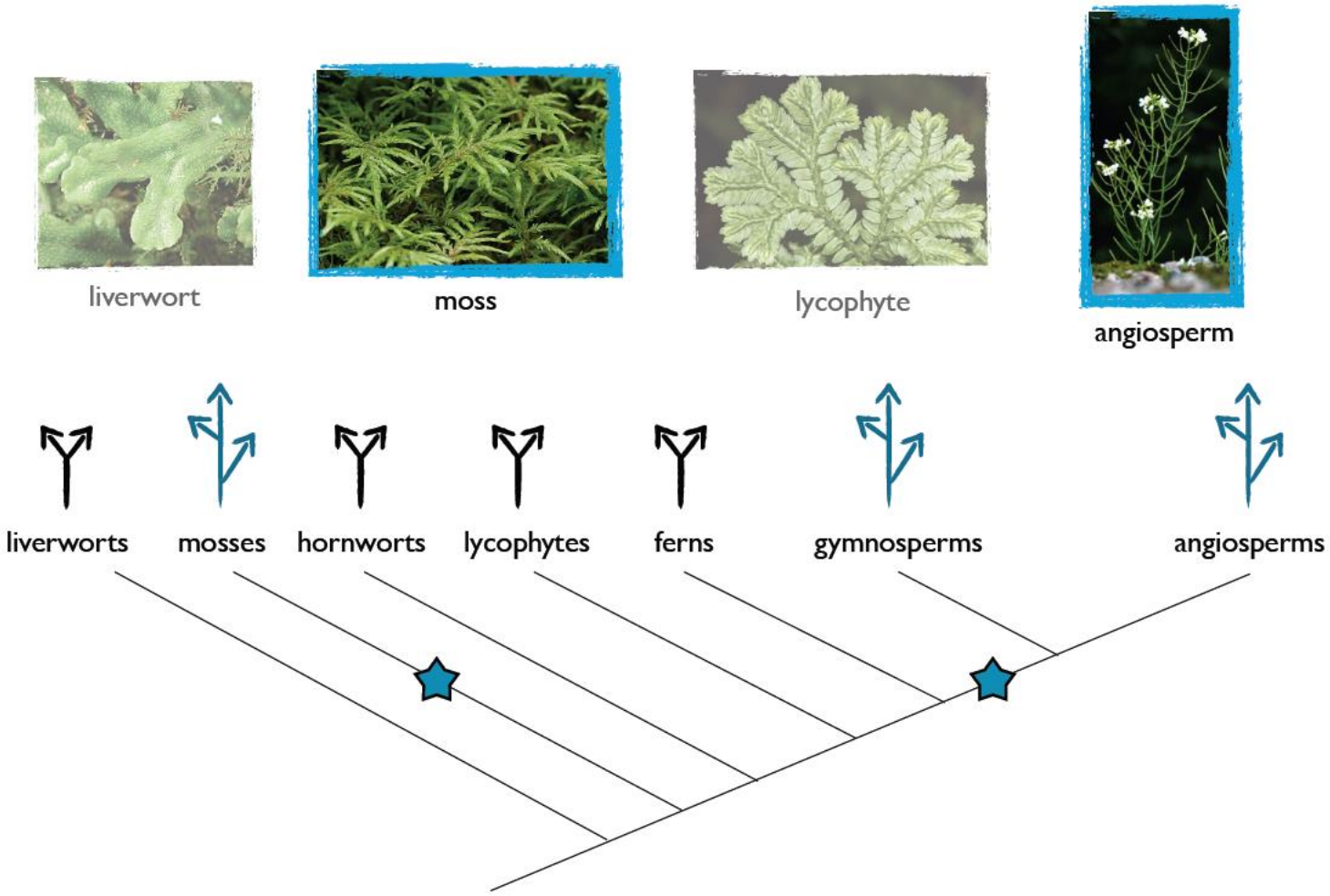


Example: how is lateral branching controlled in Physcomitrella patens?

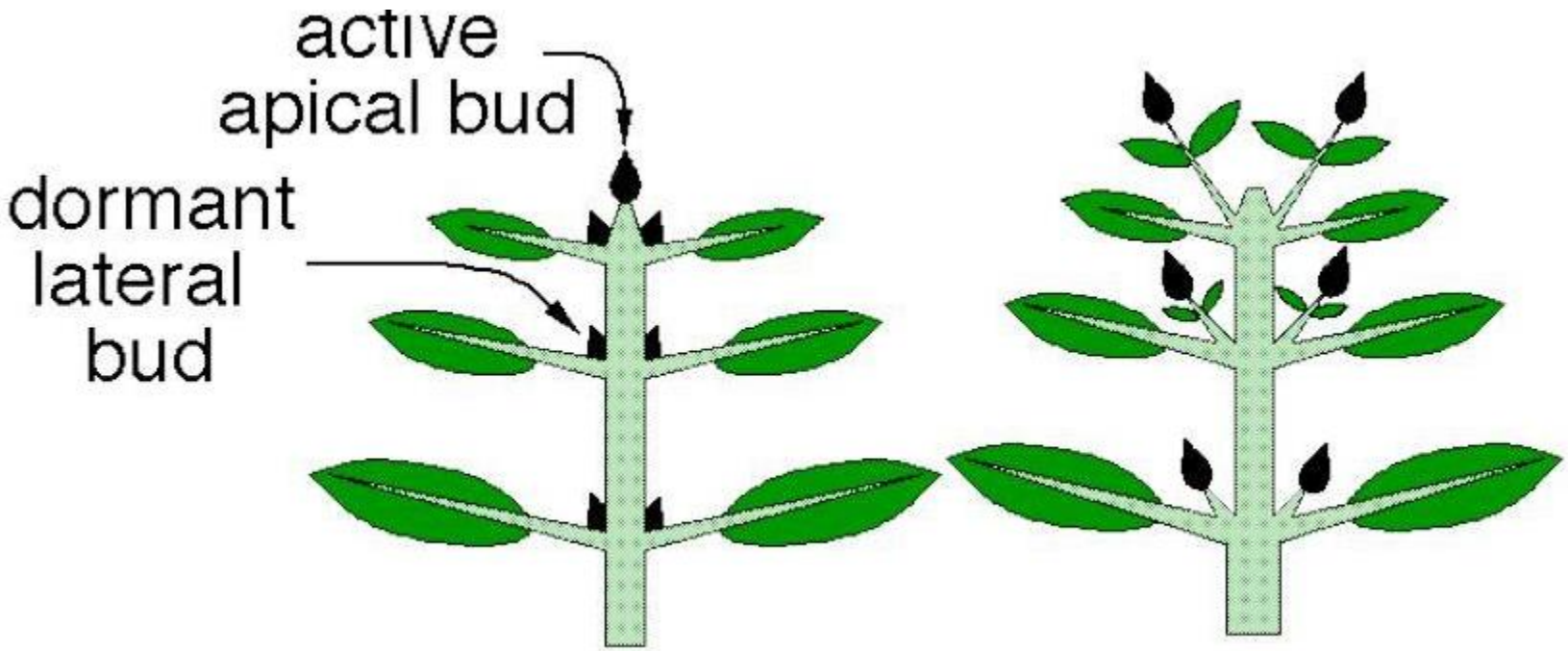


Coudert and Palubicki et al. Three ancient hormonal cues co-ordinate shoot branching in a moss. eLife 2015

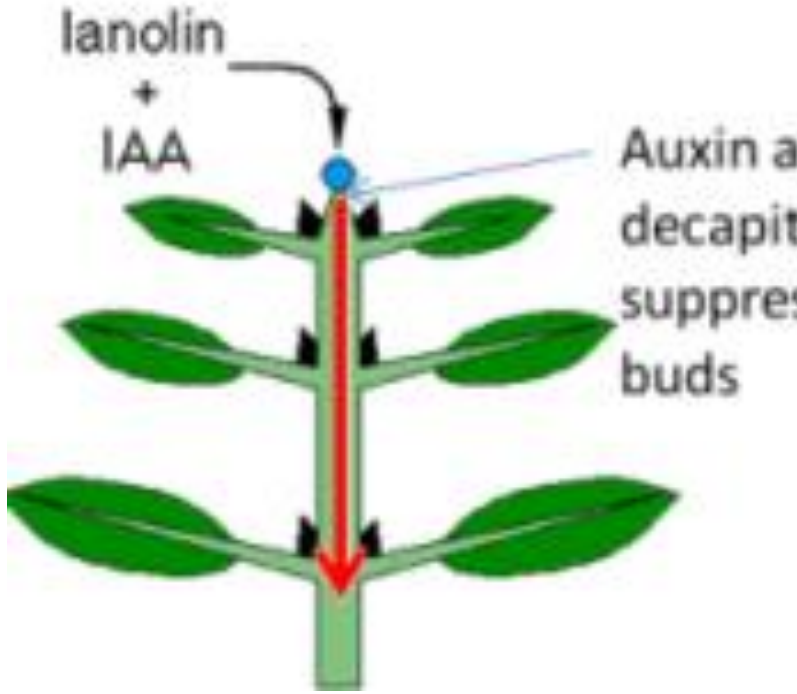
Independent evolution of lateral branching



Apical Dominance



Auxin suppresses lateral buds



Apical dominance without PAT?

Decapitation experiment performed in the moss *Splachnum ampullaceum*

Treatment	Percentage of bud reactivation
Intact	0
Decapitated + water	84
Decapitated + IAA (1 mgm./ml.)	0

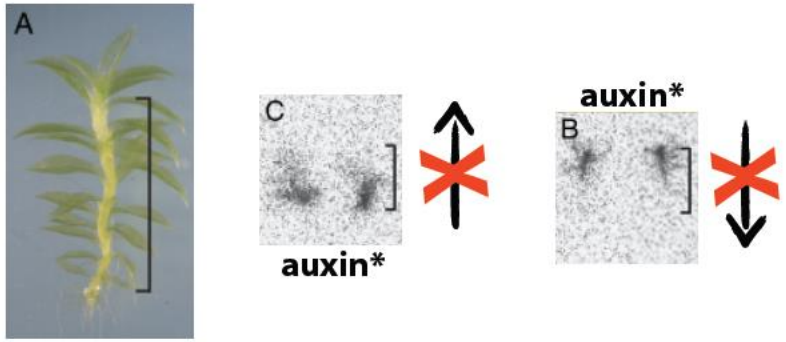
von Maltzhan et al., Nature 1959

Apical dominance without PAT?

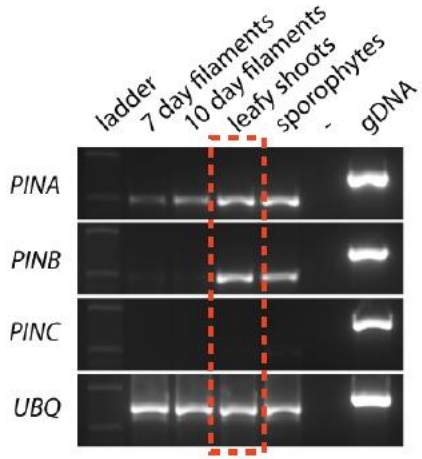
Decapitation experiment performed in the moss *Splachnum ampullaceum*

Treatment	Percentage of bud reactivation
Intact	0
Decapitated + water	84
Decapitated + IAA (1 mgm./ml.)	0

von Maltzhan et al., Nature 1959

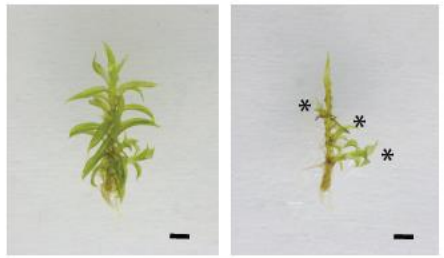


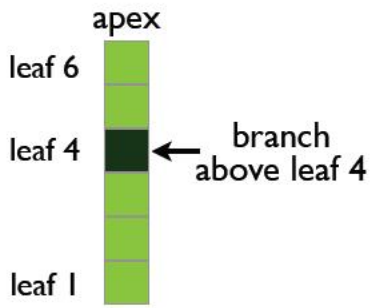
Fujita et al., Evol. Dev. 2008

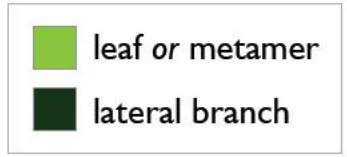


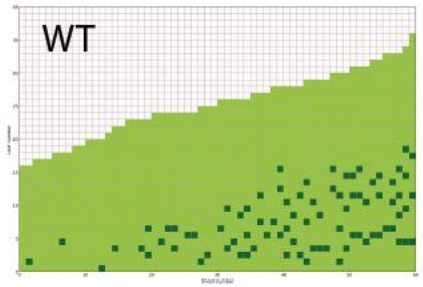
Bennett et al., in prep

Branching inhibited by the apex

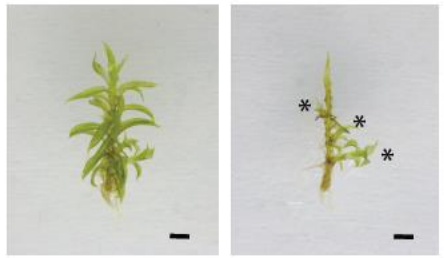


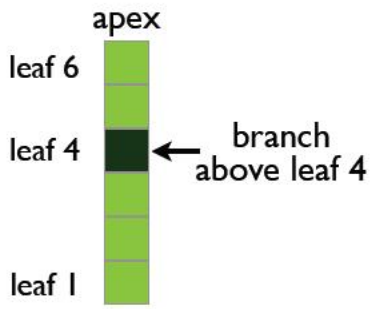


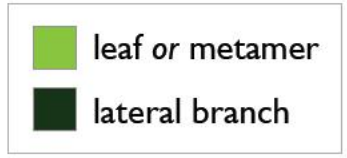


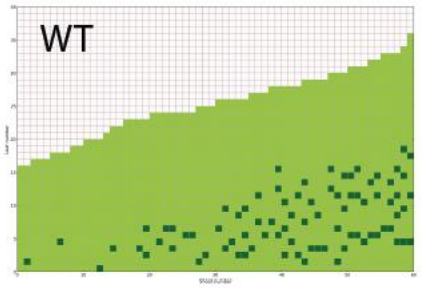


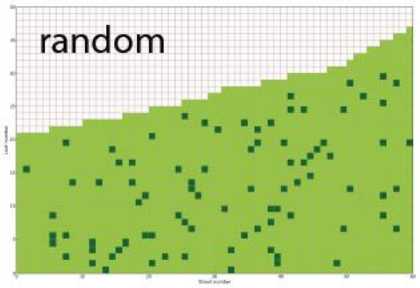
Branching inhibited by the apex



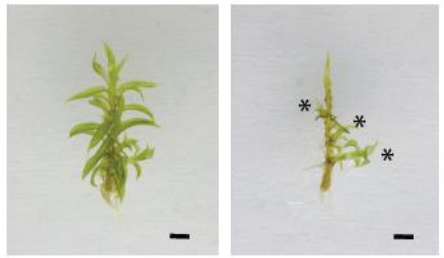


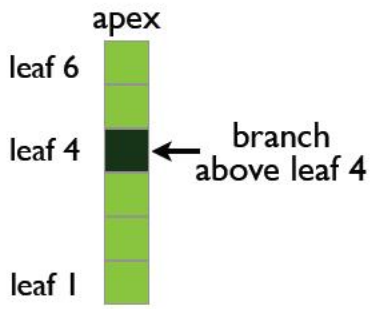


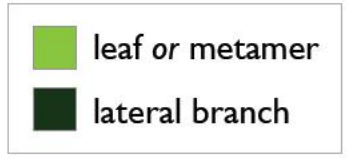


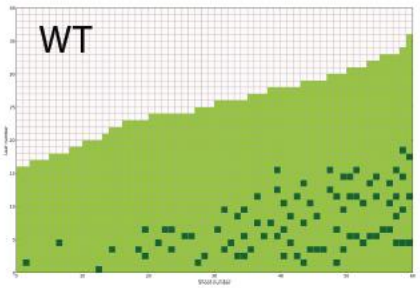


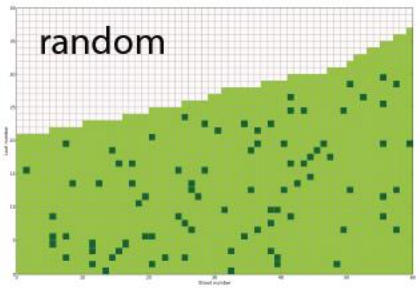
Branching inhibited by the apex

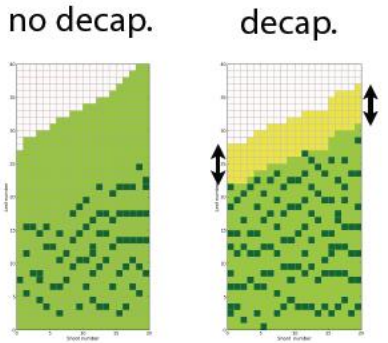






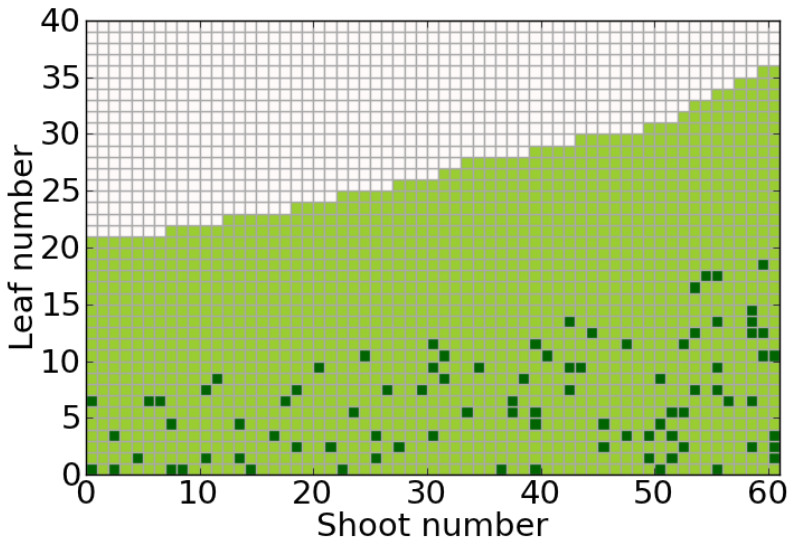




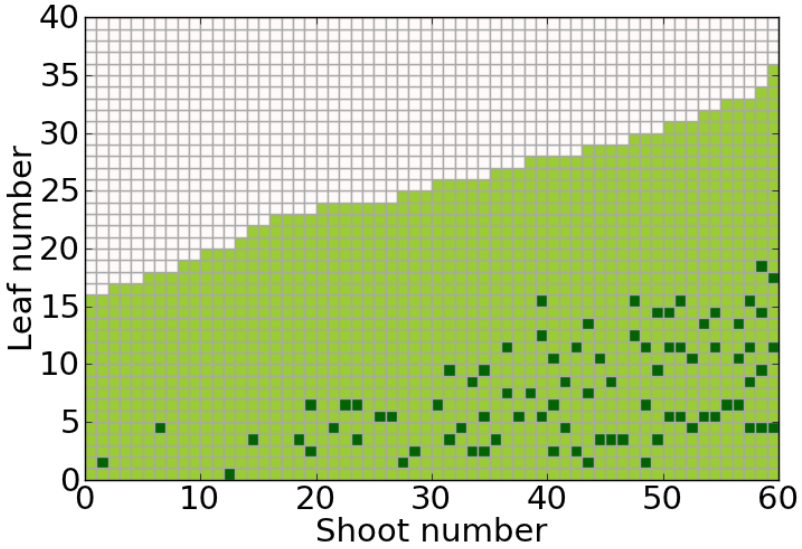


branching pattern after 1 week

Average distance between branches



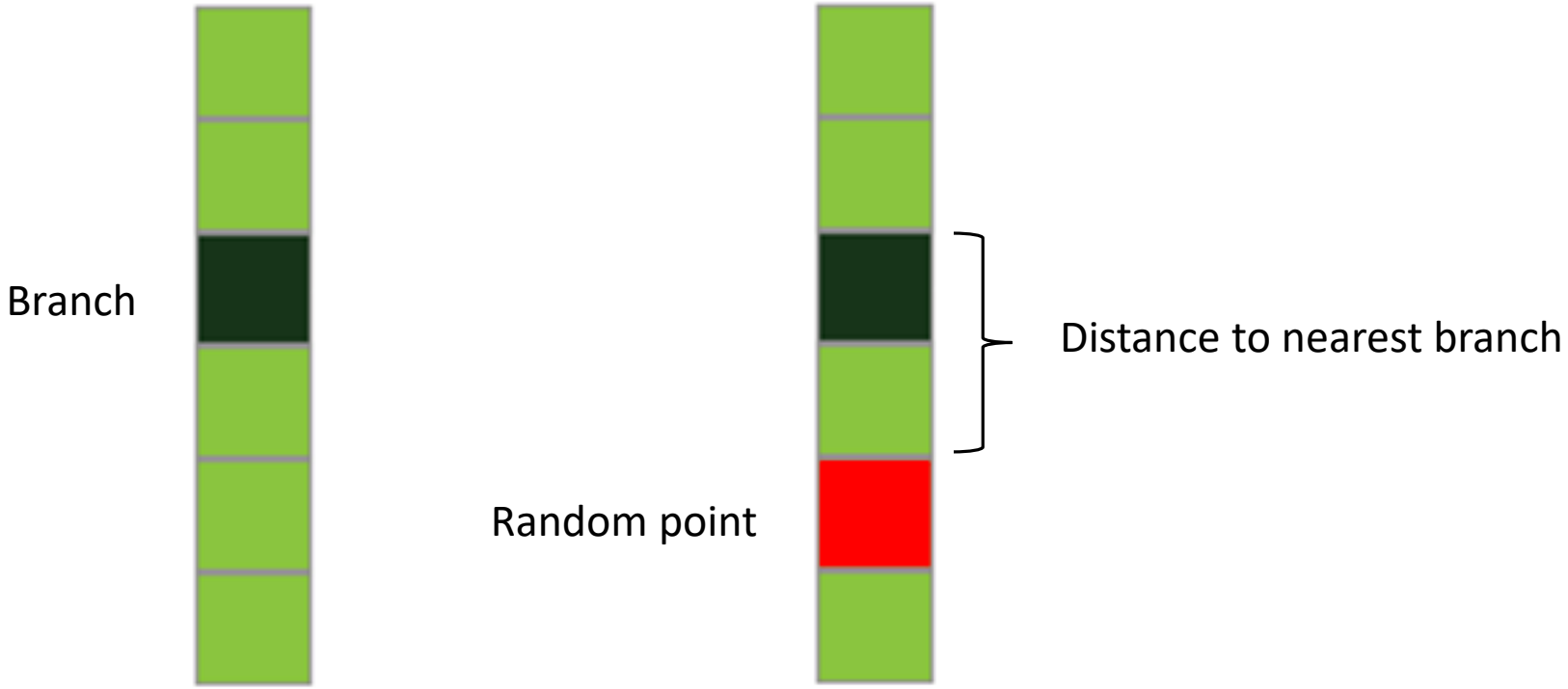
Random model:
Ave. distance between neighbors: 3.13



Wild-type:
Ave. distance between neighbors: 4.48

Branch distribution is over-dispersed?

- Calculate minimum distance of random points in the branching zone to real branches



Branch distribution is over-dispersed

- Calculate Hopkins index:

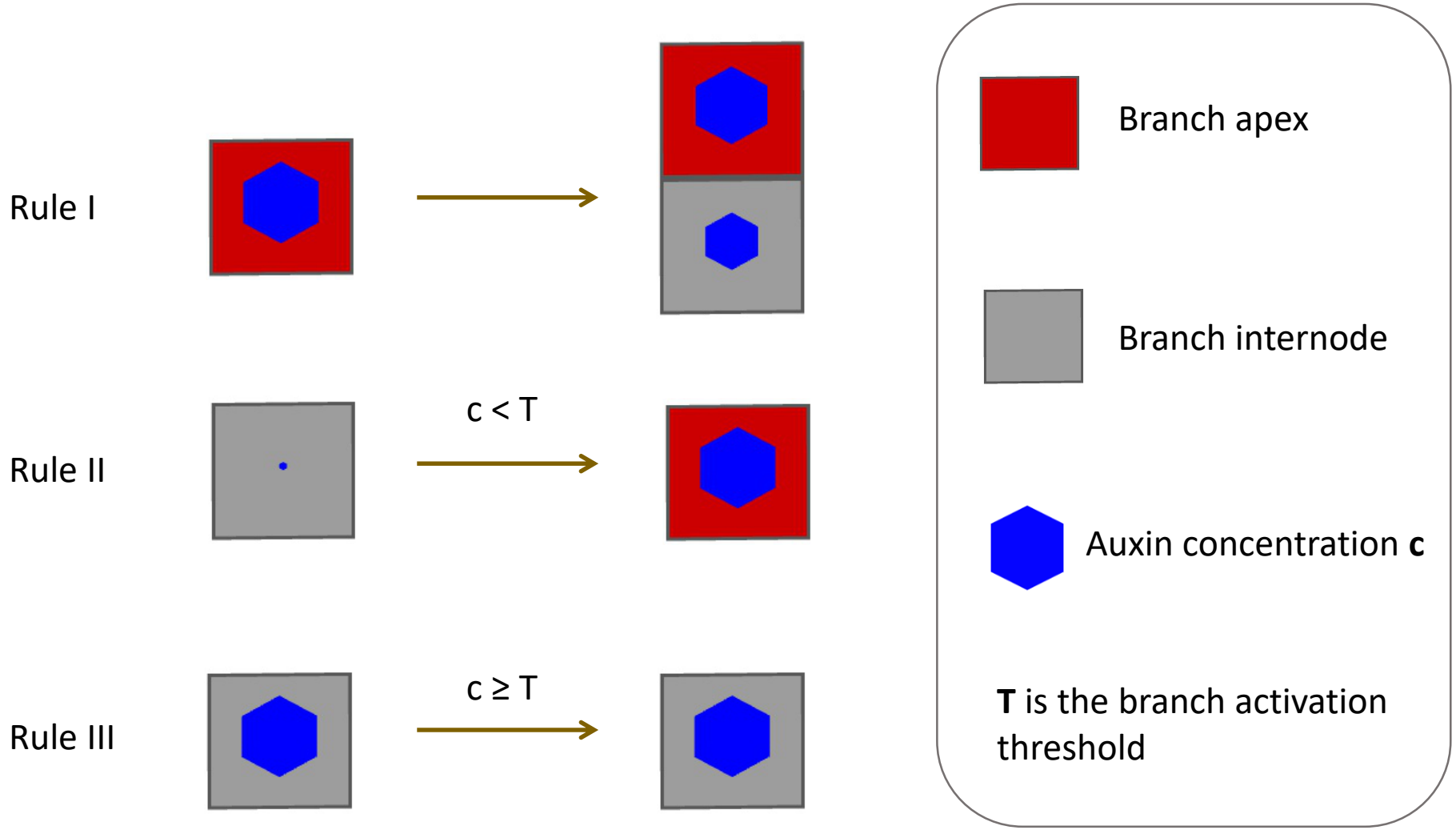
$$H = \frac{\langle min_i(\|x - b_i\|)\rangle_x}{\langle min_i(\|b_j - b_i\|)\rangle_j}$$

Wild-type data Hopkins Index: **0.81** (σ = 0.09)

Moss Branching Model

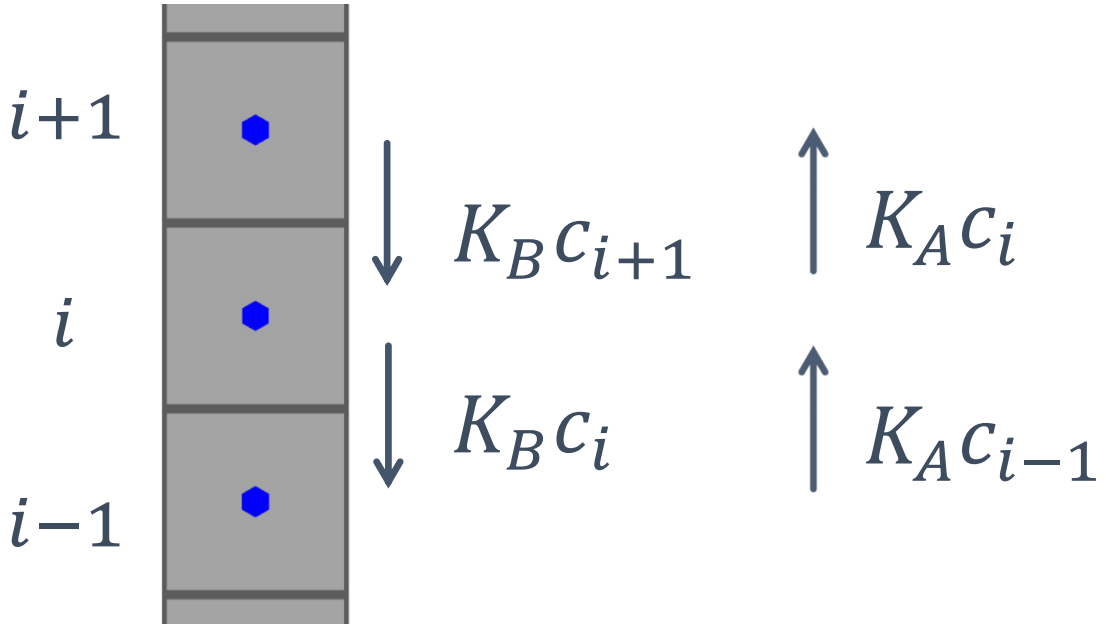
- Apices are sources of auxin production
- Branch initiation is controlled by auxin concentration
- How is auxin moving through the moss shoot?

Development in the Moss Branching Model



Directional movement of auxin in the moss branching model

$$\frac{dc_i}{dt} = K_B(c_{i+1} - c_i) + K_A(c_{i-1} - c_i) - \nu c_i$$

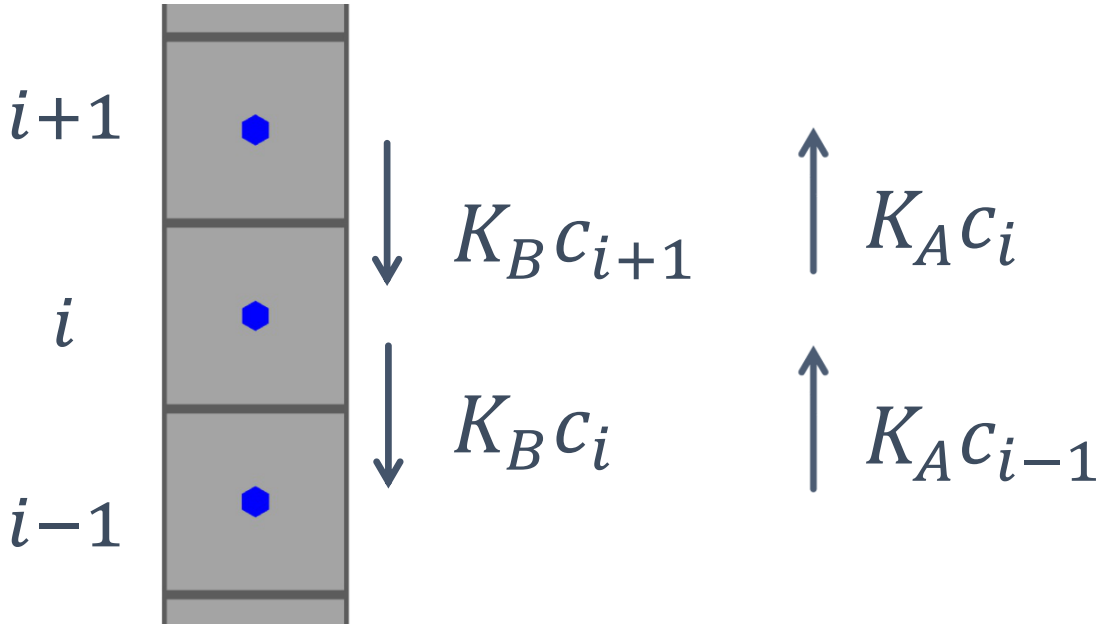


Directional movement of auxin in the moss branching model

$$\frac{dc_i}{dt} = K_B(c_{i+1} - c_i) + K_A(c_{i-1} - c_i) - \nu c_i$$

$K_B \approx K_A$

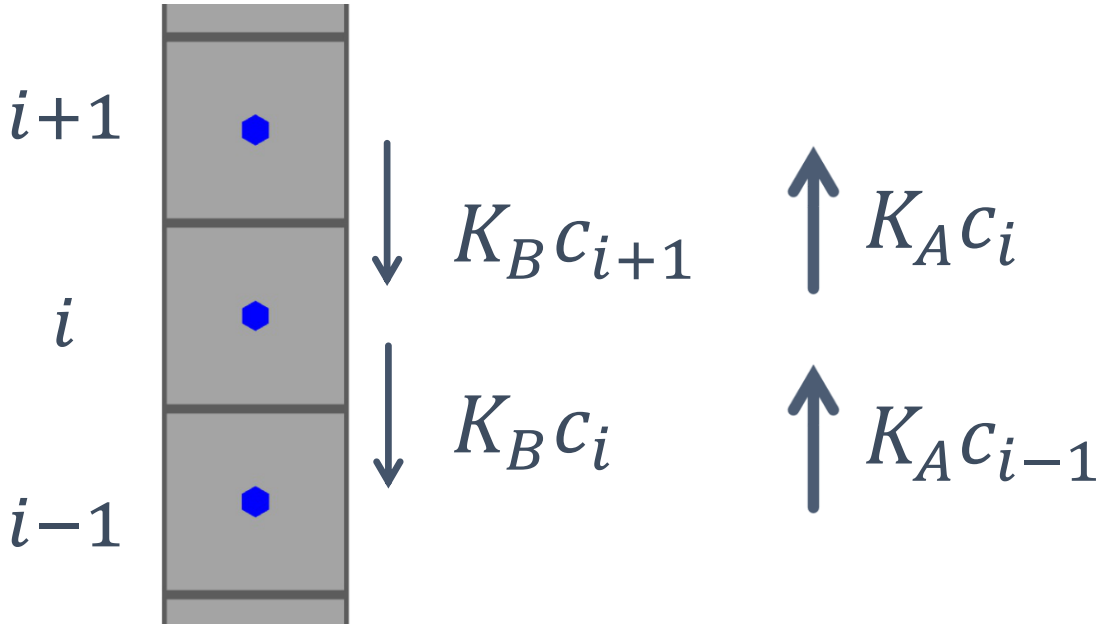
Diffusion



Directional movement of auxin in the moss branching model

$$\frac{dc_i}{dt} = K_B(c_{i+1} - c_i) + K_A(c_{i-1} - c_i) - \nu c_i$$

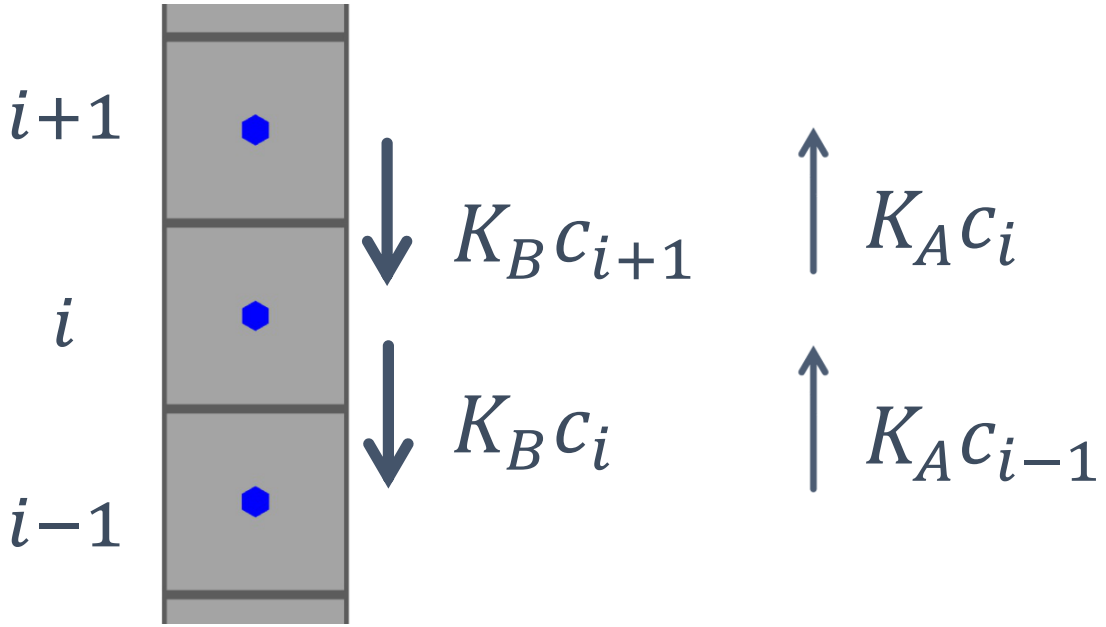
$K_B \ll K_A$



Directional movement of auxin in the moss branching model

$$\frac{dc_i}{dt} = K_B(c_{i+1} - c_i) + K_A(c_{i-1} - c_i) - \nu c_i$$

$K_B \gg K_A$

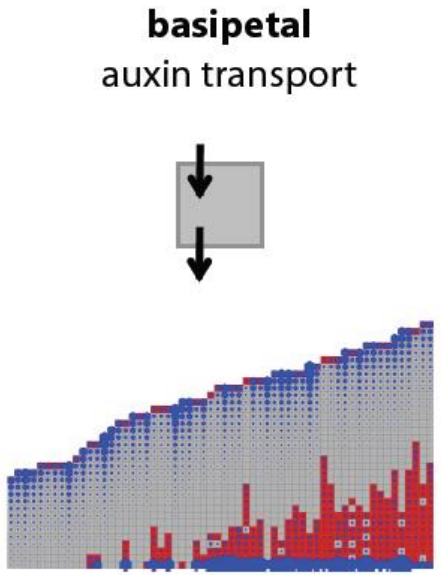


Input Parameters

- H_{apex} is the target auxin concentration of the terminal apex (stochastic parameter)
- H is the target auxin concentration of lateral apices (stochastic parameter)
- T is the branching threshold (stochastic parameter)
- ν is the auxin concentration decay rate
- K_A and K_B determine the rate of directional auxin transport

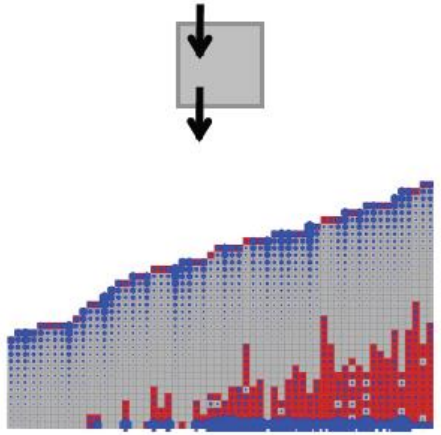
Development and Auxin Transport

Directional movement of auxin

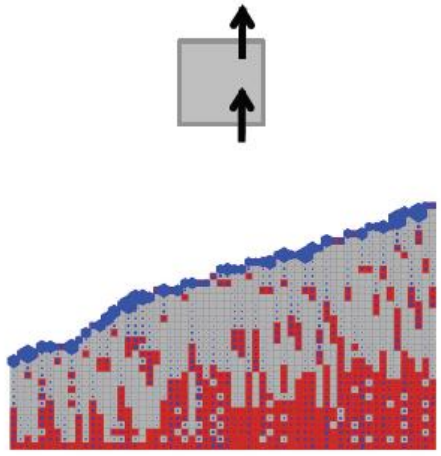


Directional movement of auxin

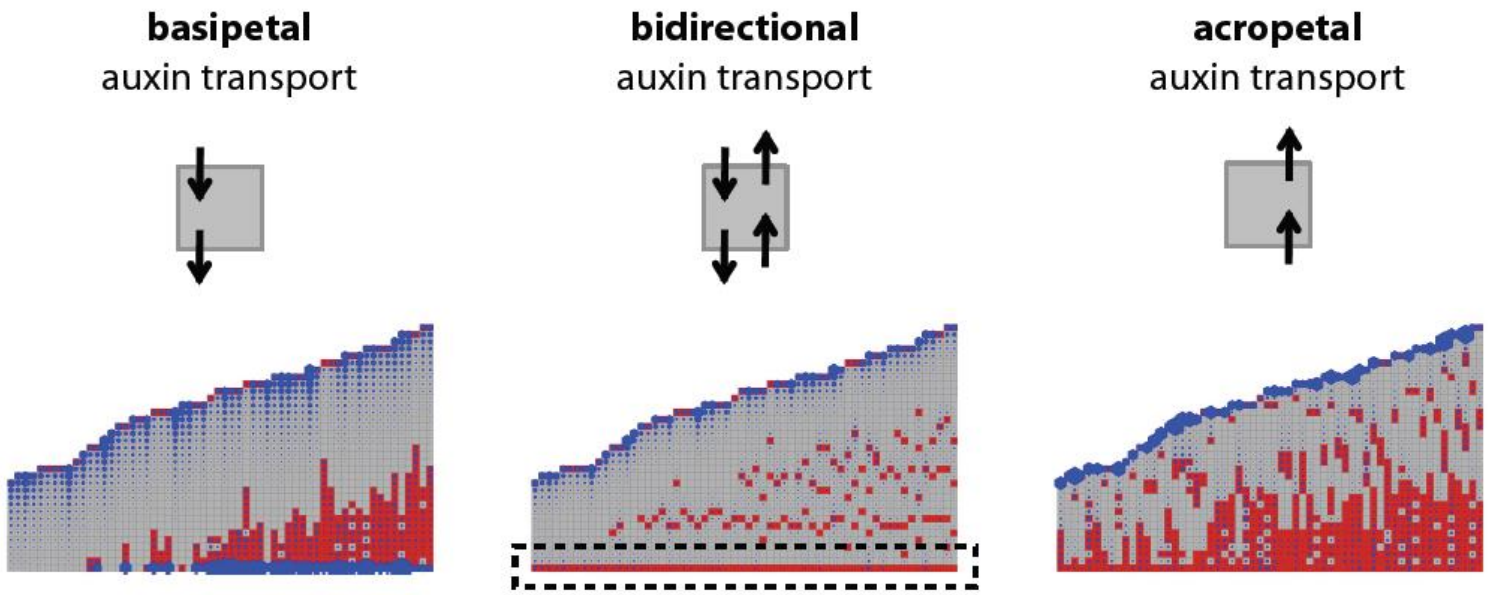
basipetal
auxin transport



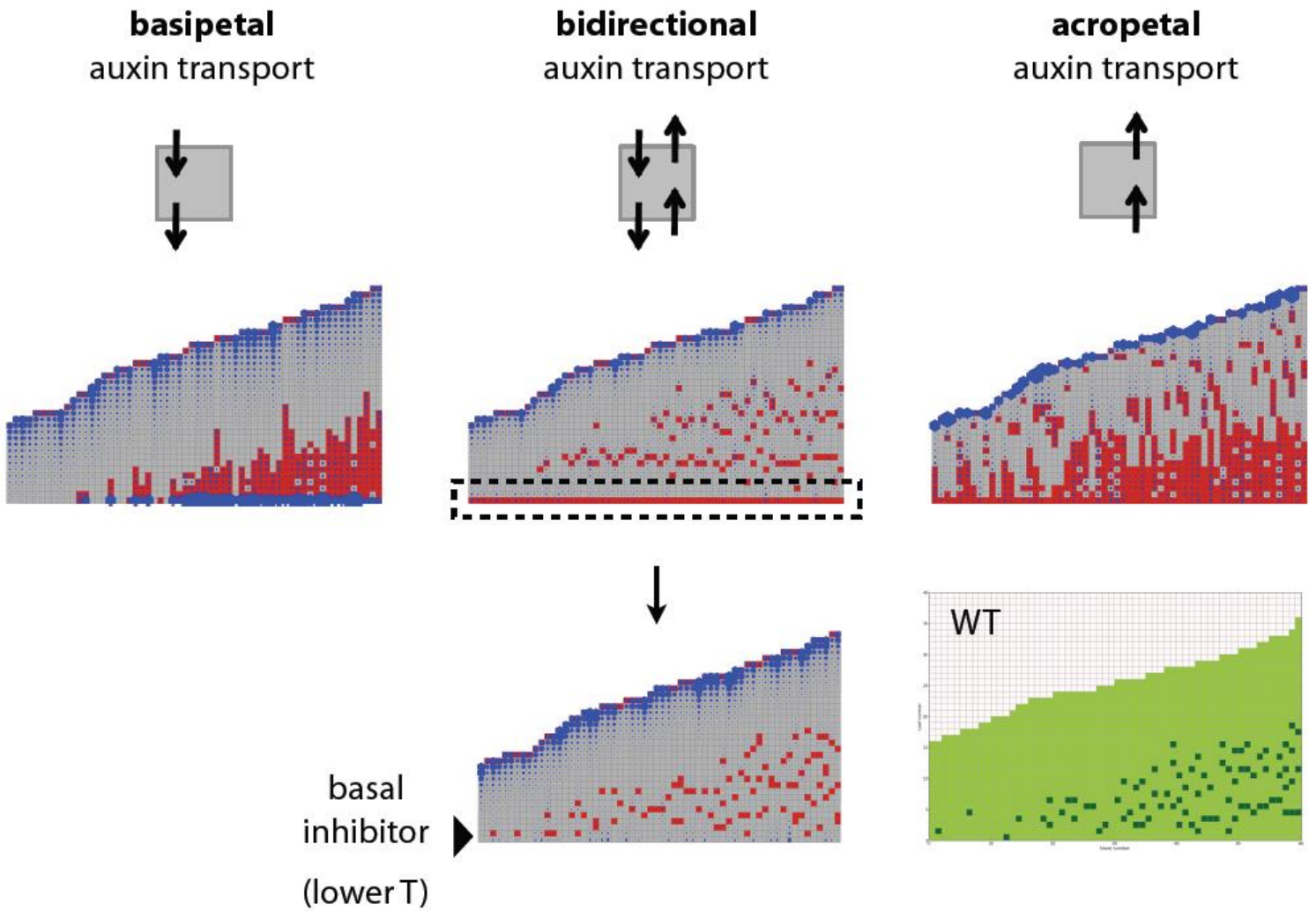
acropetal
auxin transport



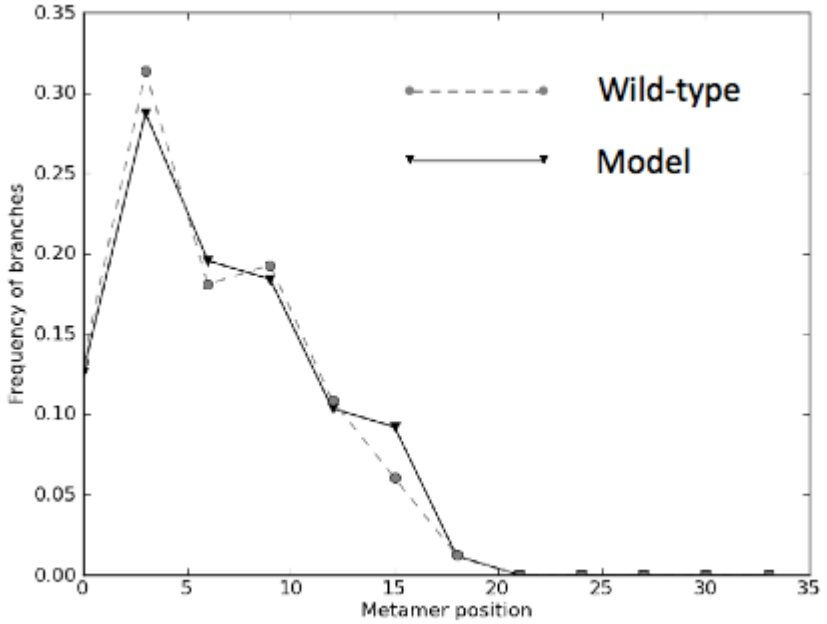
Directional movement of auxin

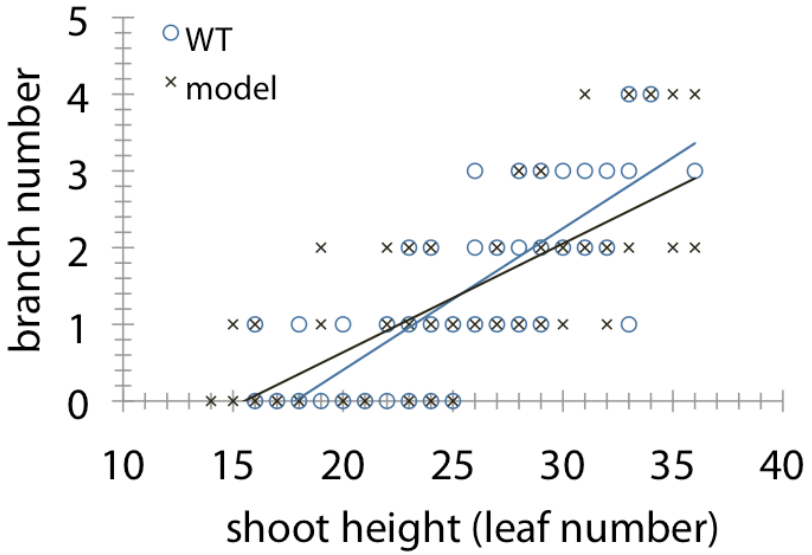


Bidirectional transport model captures real data



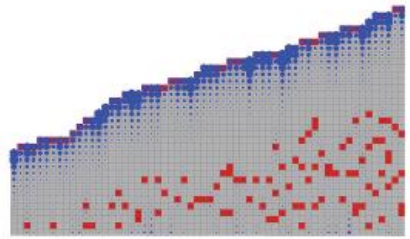
Quantitative analysis: bidirectional model

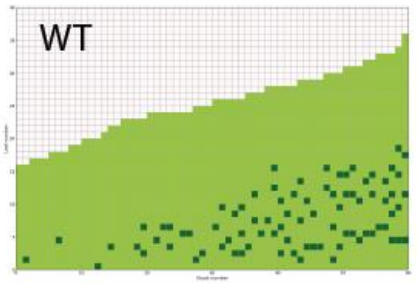


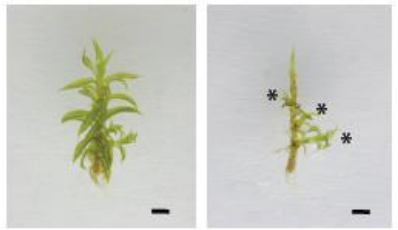


	Model	Wild-type
Branch number	87	83
Distance between branches	4.46	4.48
Apical inhibition zone	18.0	17.9

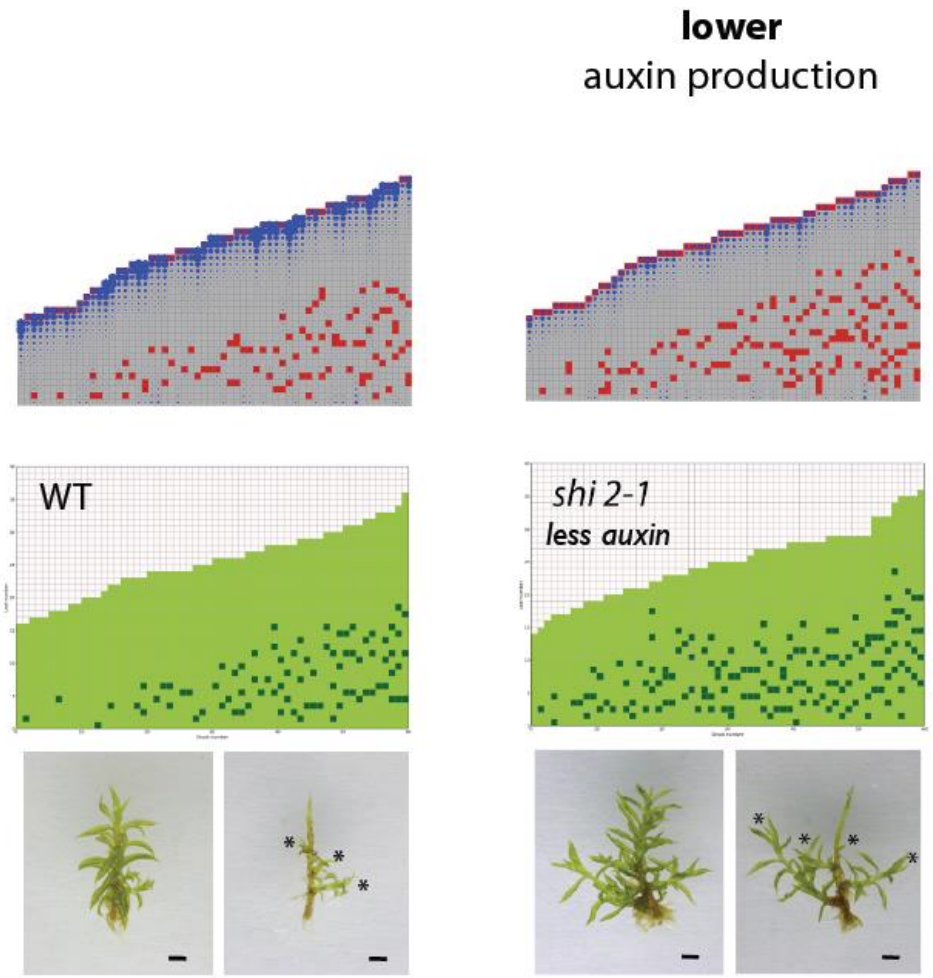
Branching patterns in altered auxin level mutants



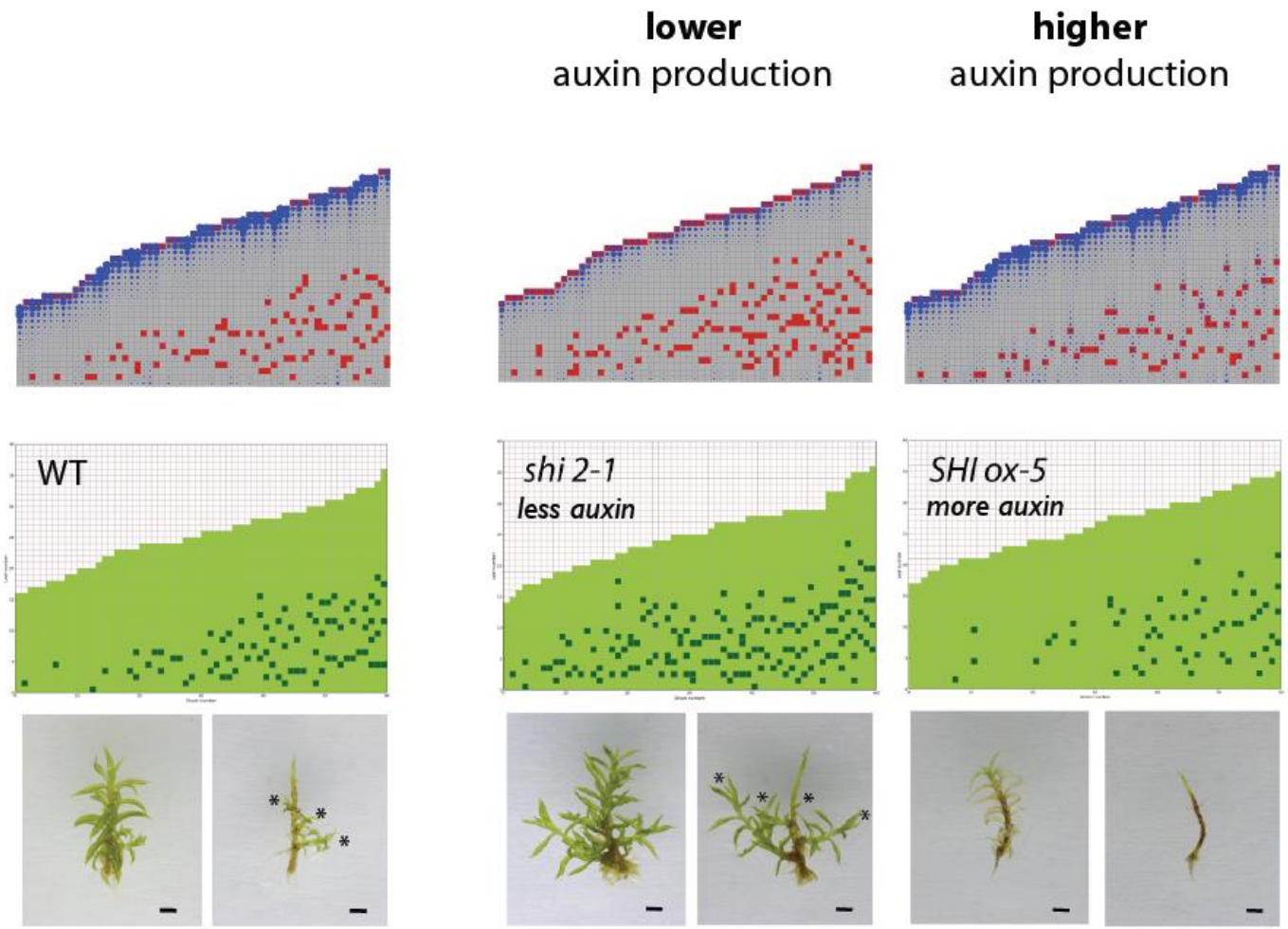




Branching patterns in altered auxin level mutants



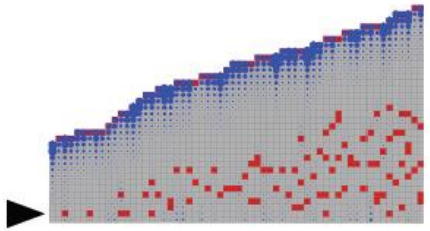
Branching patterns in altered auxin level mutants



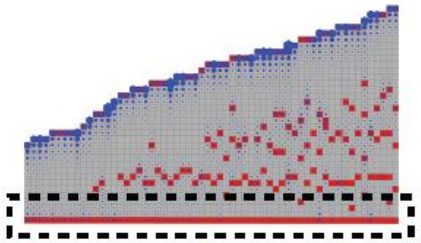
Strigolactone deficient mutant

WT with basal inhibitor

lower basal threshold



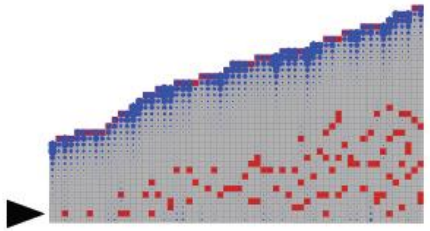
WT without basal inhibitor



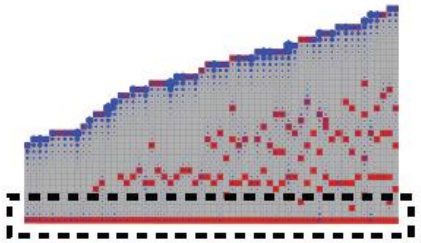
Strigolactone deficient mutant

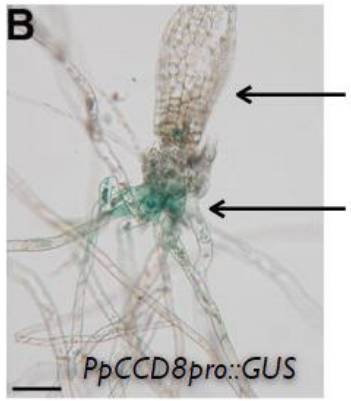
WT with basal inhibitor

lower basal threshold

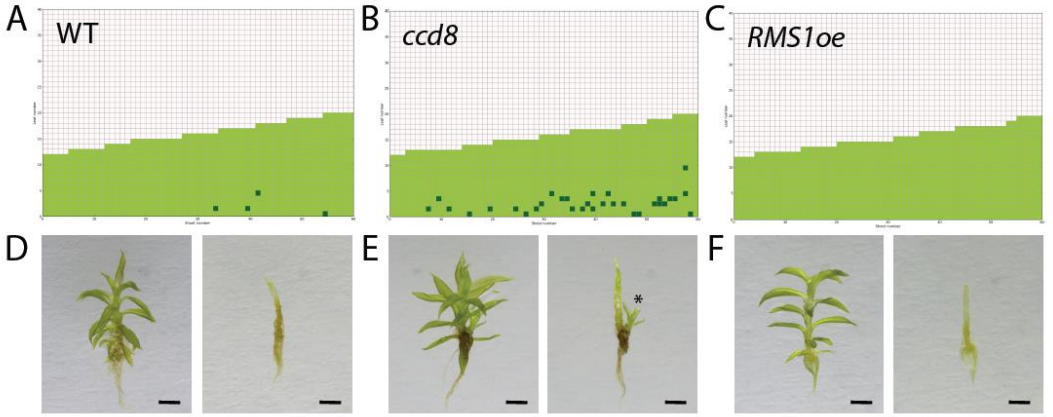


WT without basal inhibitor

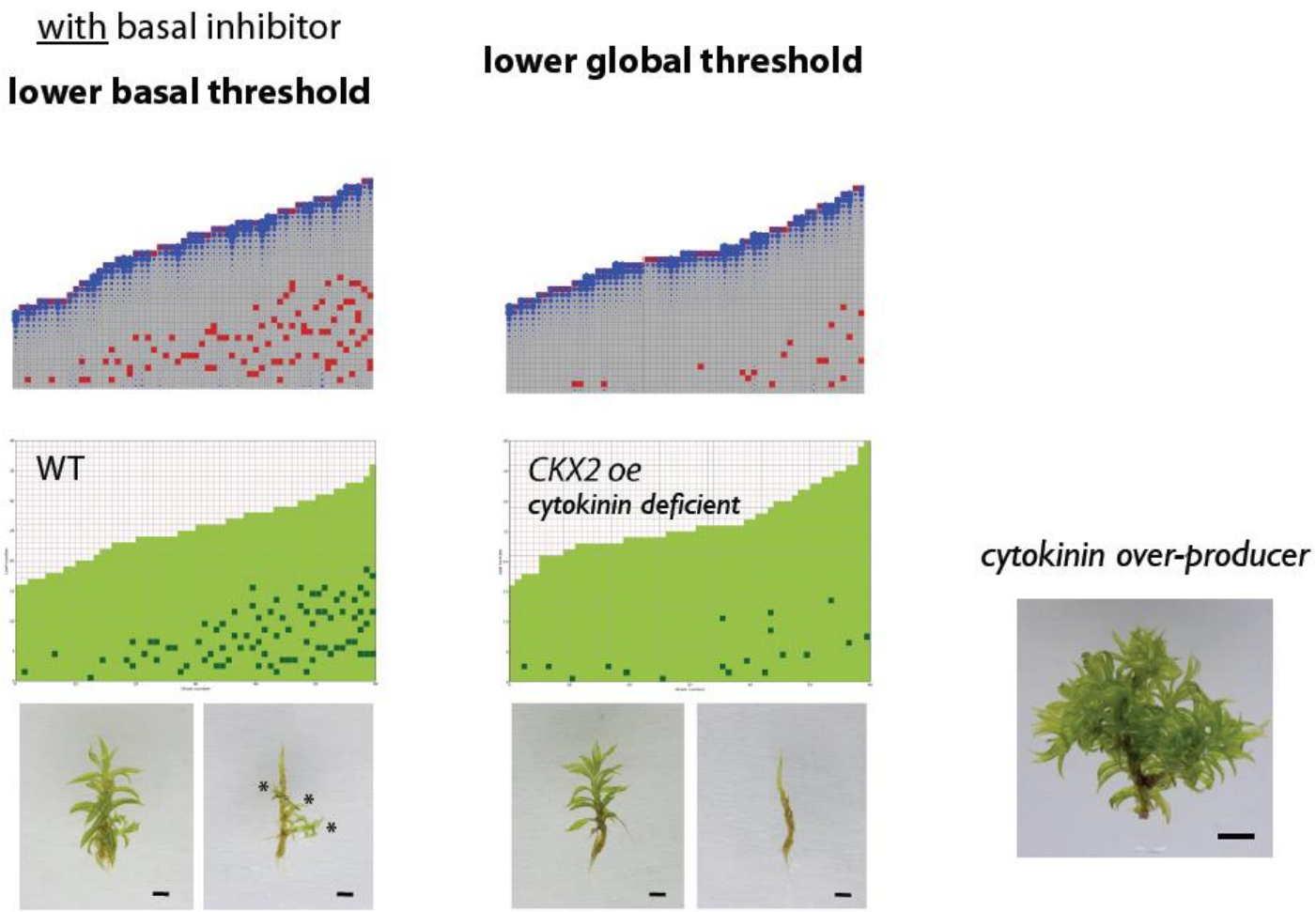




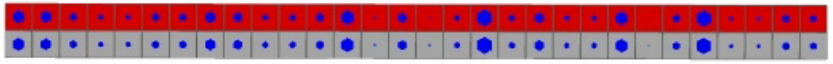
Proust *et al.*, *Development* 2011



Cytokinin deficient mutant

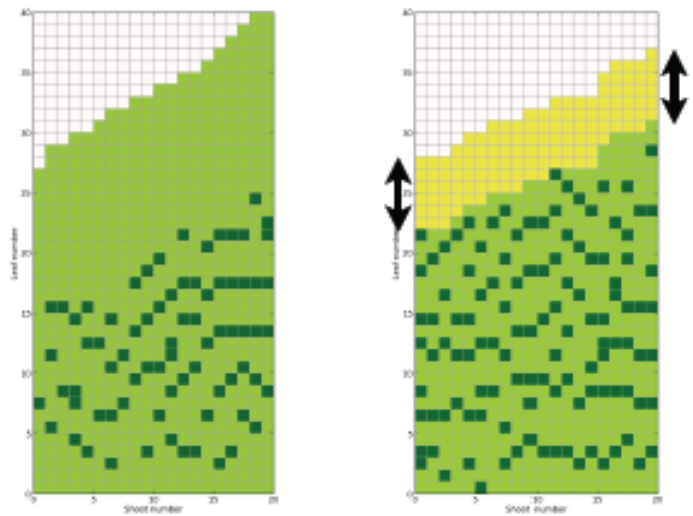


Decapitation simulation

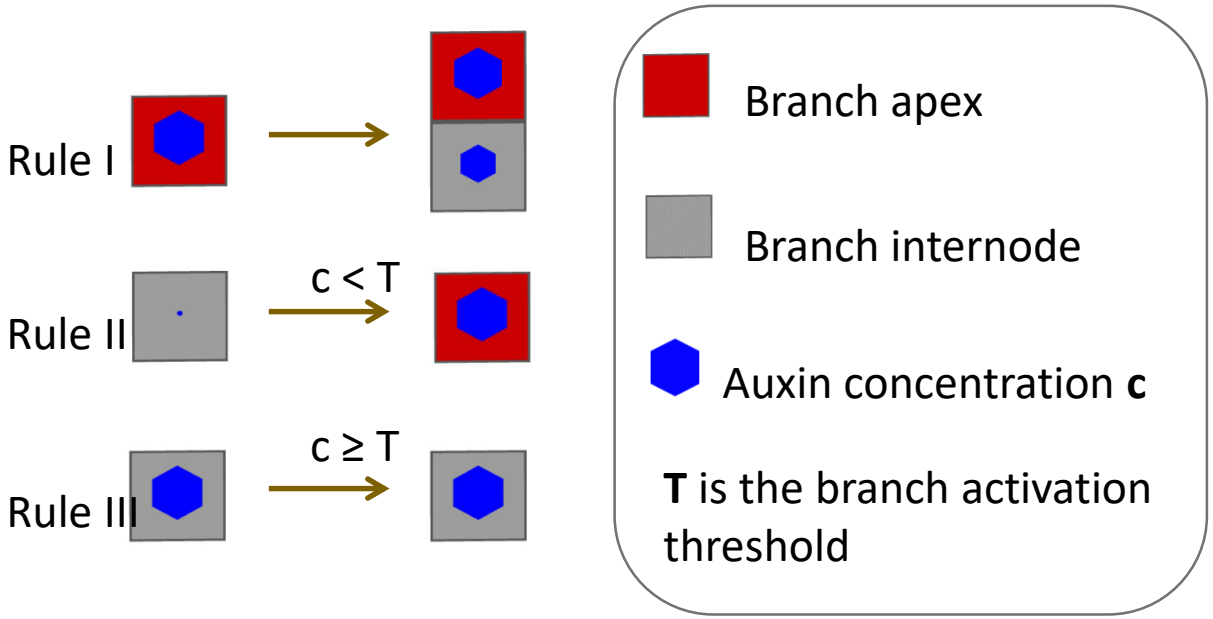


no decap.

decap.



branching pattern
after 1 week

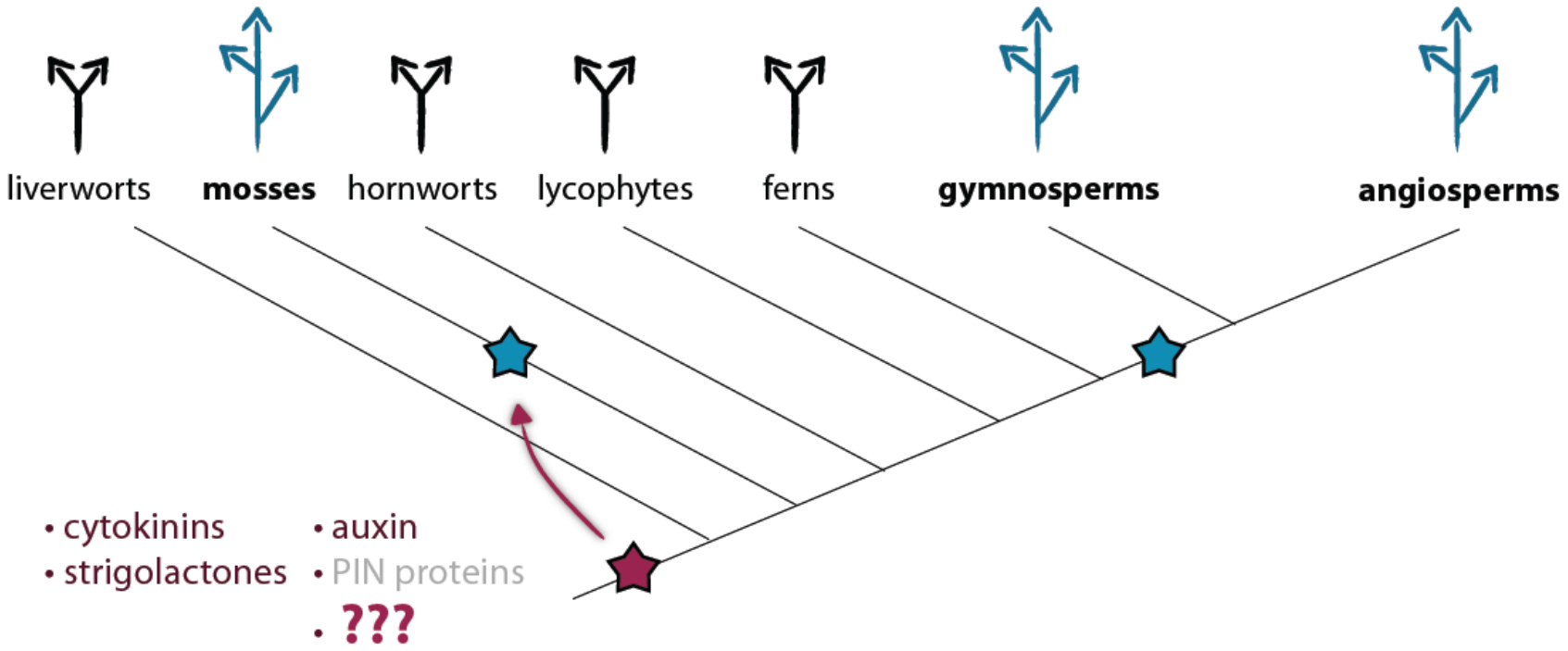


$$\frac{dc_i}{dt} = K_B(c_{i+1} - c_i) + K_A(c_{i-1} - c_i) - \nu c_i$$

Conclusion

- Auxin, cytokinin and strigolactones control major aspects of branching patterns in *Physcomitrella patens*
- Parameter space of simple model captures branching patterns of mutant and transgenic lines:
  - Denser branching at low auxin levels
  - Sparser branching at high auxin levels
  - Local and global thresholds of branching capture effects of cytokinin and strigolactones

Conclusion


liverworts
mosses
hornworts
lycophytes
ferns
gymnosperms
angiosperms
• cytokinins
• strigolactones
• auxin
• PIN proteins
• ???

2D Numpy array

```
U = np.random.rand(size, size)
```

Metoda różnic skończonych

$$\Delta u(x,y) \simeq \frac{u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)}{dx^2}$$

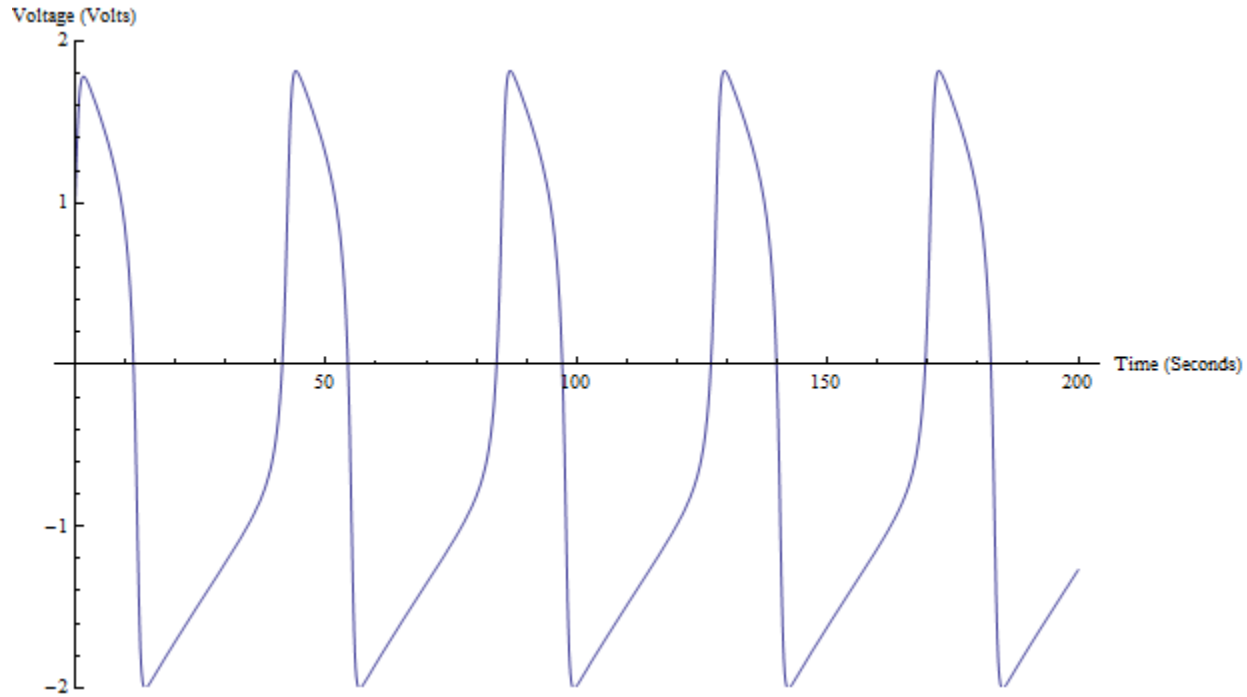
Implementacja za pomocy numpy arrays

```
def aproksymacjaL(X):
    XGora = X[1:-1,2:]
    XDol = X[1:-1,0:-2]
    XLewo = X[0:-2,1:-1]
    XPrawo = X[2:,1:-1]
    XSrodek = X[1:-1,1:-1]
    return (XGora + XDol + XLewo + XPrawo - 4 * XSrodek)
```

Zadanie

- Rozszerz model z poprzedniego zadania do dwóch wymiarów.
- Stwórz za pomocy modułu matplotlib wykres konturowy koncentracji substancji w kilka punktach czasowych.

Analiza i prezentacja modelu neuronu: FitzHugh-Nagumo



Wzrost Anabaena, L-system + rown. rozn.

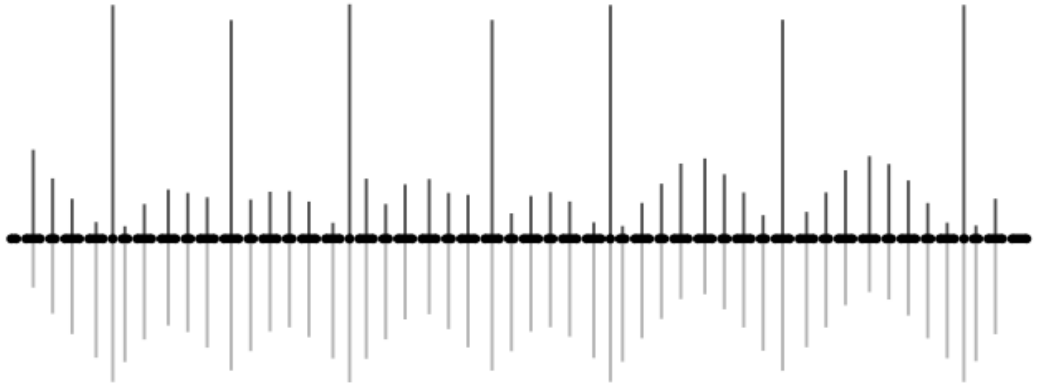


Figure 3: Fragment of a simulated filament of *Anabaena*. Vertical lines indicate the concentrations of the activator and inhibitor (above and below the cells, respectively). Notice the sharp peaks of the activator concentration that define the heterocysts, and high levels of the inhibitor concentration in the neighboring vegetative, which prevent their differentiation. The parameters used in the simulation were: $\rho = 3$, $\kappa = 0.001$, $a_0 = 0.01$, $\mu = 0.1$, $h_0 = 0.001$, $\nu = 0.45$, $D_h = 0.004$, $a_{th} = 1$, $k = 0.38196$, $s_{max} = 1$, $r = 0.002$, and $w = 0.001$.

Mark Hammel and Przemyslaw Prusinkiewicz. L-systems and partial differential equations.

Rozwój starożytnych cywilizacji

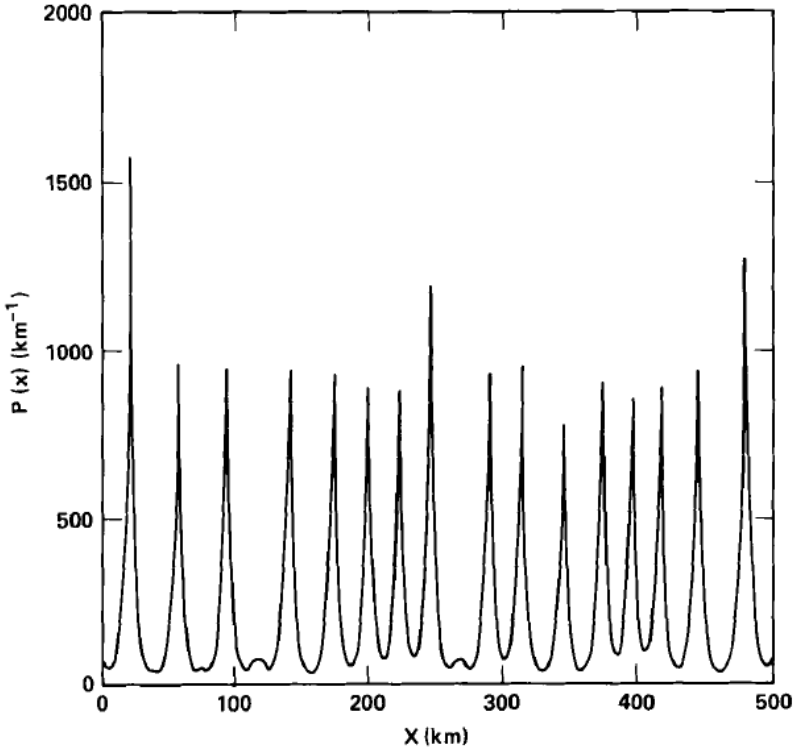
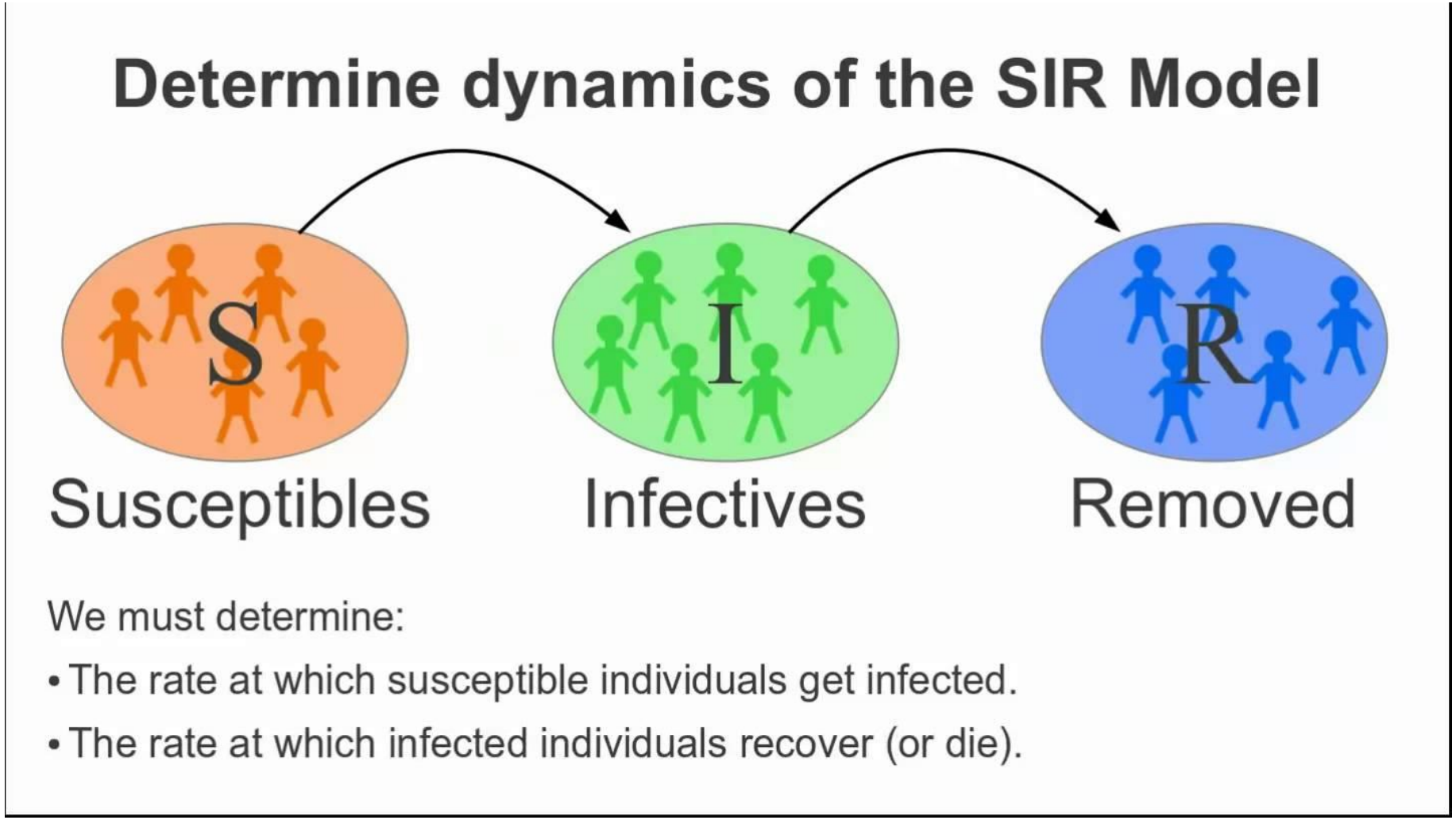


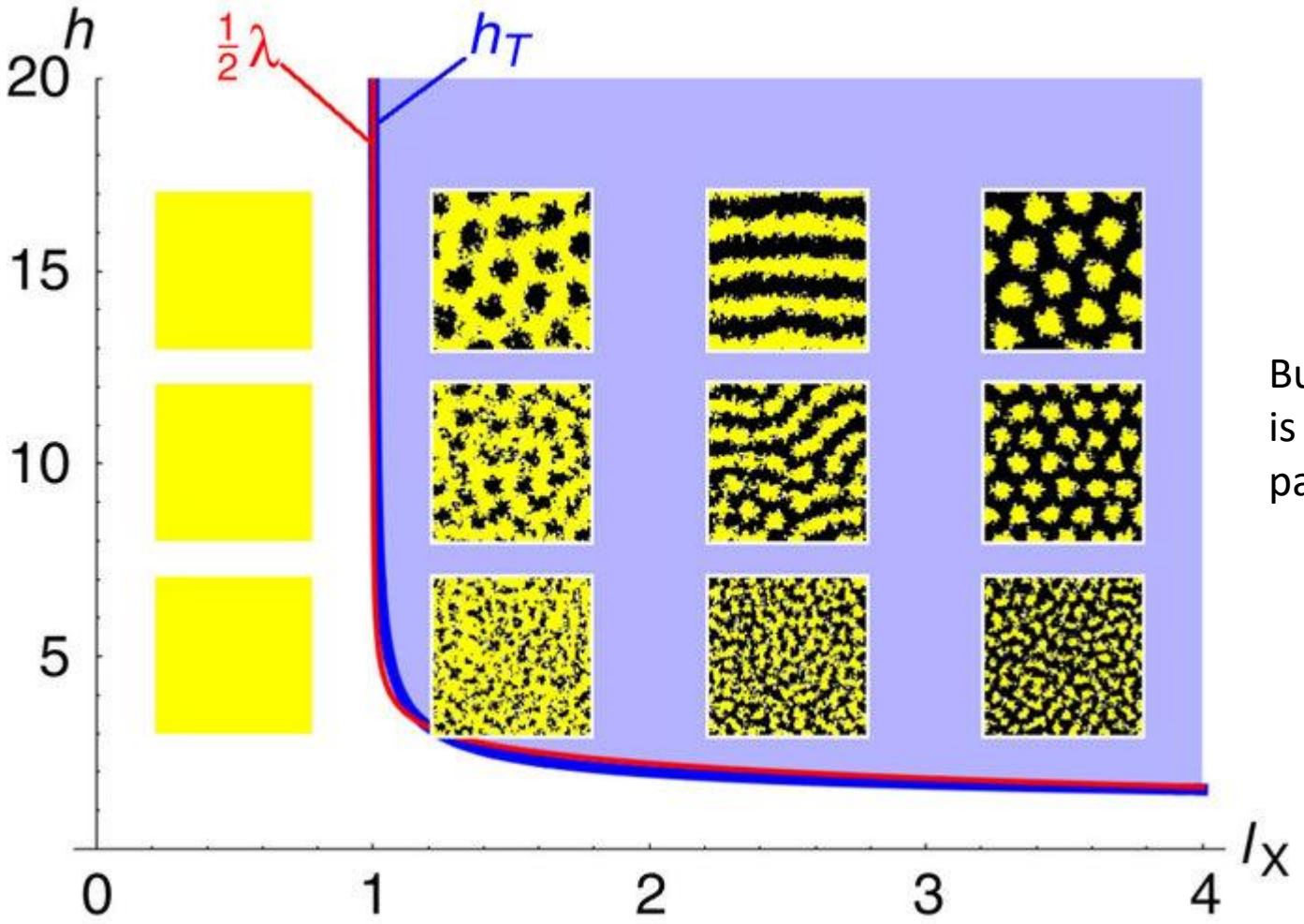
FIG. 5. A typical population profile $P(x)$ of the social system. The cities are located at the sharp spikes, and the single state is centered in the city at $x = 21$ km.

Young. A spatiotemporal model of civilization.

Prezentacja Modeli epidemicznych

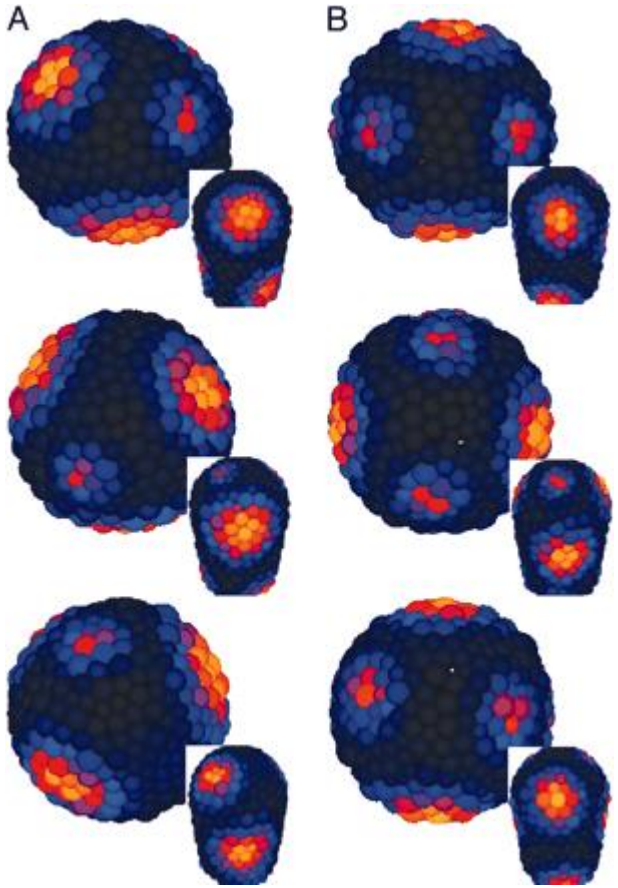


Modelowanie Pasek ryb



Bullara, Decker. Pigment cell movement is not required for generation of Turing patterns in zebrafish skin

Modelowanie mechanizmów kontrolujące filotaksje

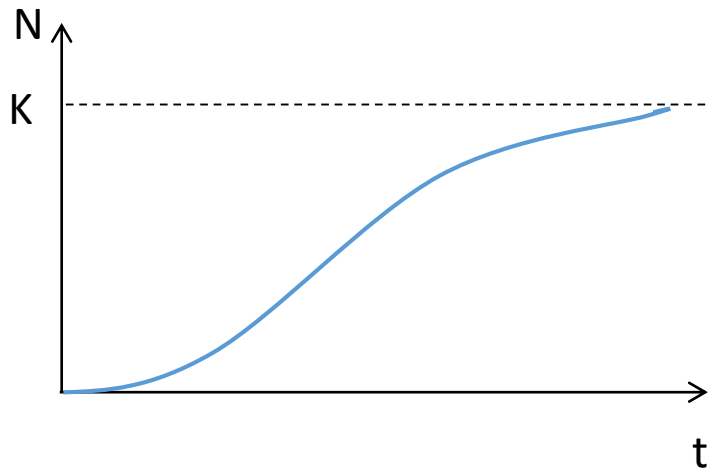


Jonsson et al.. An auxin-driven polarized transport model for phyllotaxis

Model Logistyczny

- Wzrost populacji N

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = rN\frac{K-N}{K}$$



K → pojemność środowiska

r → współczynnik wzrostu

Model Kompetycji Lotka-Volterra

Kompetycja Lotka-Volterra

- Model dwóch populacji które konkurują o te same zasoby

- N_1, N_2 populacje zajęców i owiec

Kompetycja Lotka-Volterra

- Model dwóch populacji które konkurują o te same zasoby

- N_1, N_2 populacje zajęców i owiec

$$\frac{dN_1}{dt} = r_1 N_1 \frac{K_1 - N_1}{K_1}$$

K → pojemność środowiska

r → współczynnik wzrostu

Kompetycja Lotka-Volterra

- Model dwóch populacji które konkurują o te same zasoby

- N_1, N_2 populacje zajęców i owiec

$$\frac{dN_1}{dt} = r_1 N_1 \frac{K_1 - N_1 - \beta_1 N_2}{K_1}$$

K → pojemność środowiska

r → współczynnik wzrostu

Kompetycja Lotka-Volterra

- Model dwóch populacji które konkurują o te same zasoby

- N_1, N_2 populacje zajęców i owiec

$$\frac{dN_1}{dt} = r_1 N_1 \frac{K_1 - N_1 - \beta_1 N_2}{K_1}$$

$$\frac{dN_2}{dt} = r_2 N_2 \frac{K_2 - N_2 - \beta_2 N_1}{K_2}$$

K → pojemność środowiska

r → współczynnik wzrostu

Pytania

- Czy zające i owce mogą koegzystować?
- Czy jeden z gatunków zdominuje drugiego?

Pytania

- Czy zające i owce mogą koegzystować?
- Czy jeden z gatunków zdominuje drugiego?

- Nullclines:

Pytania

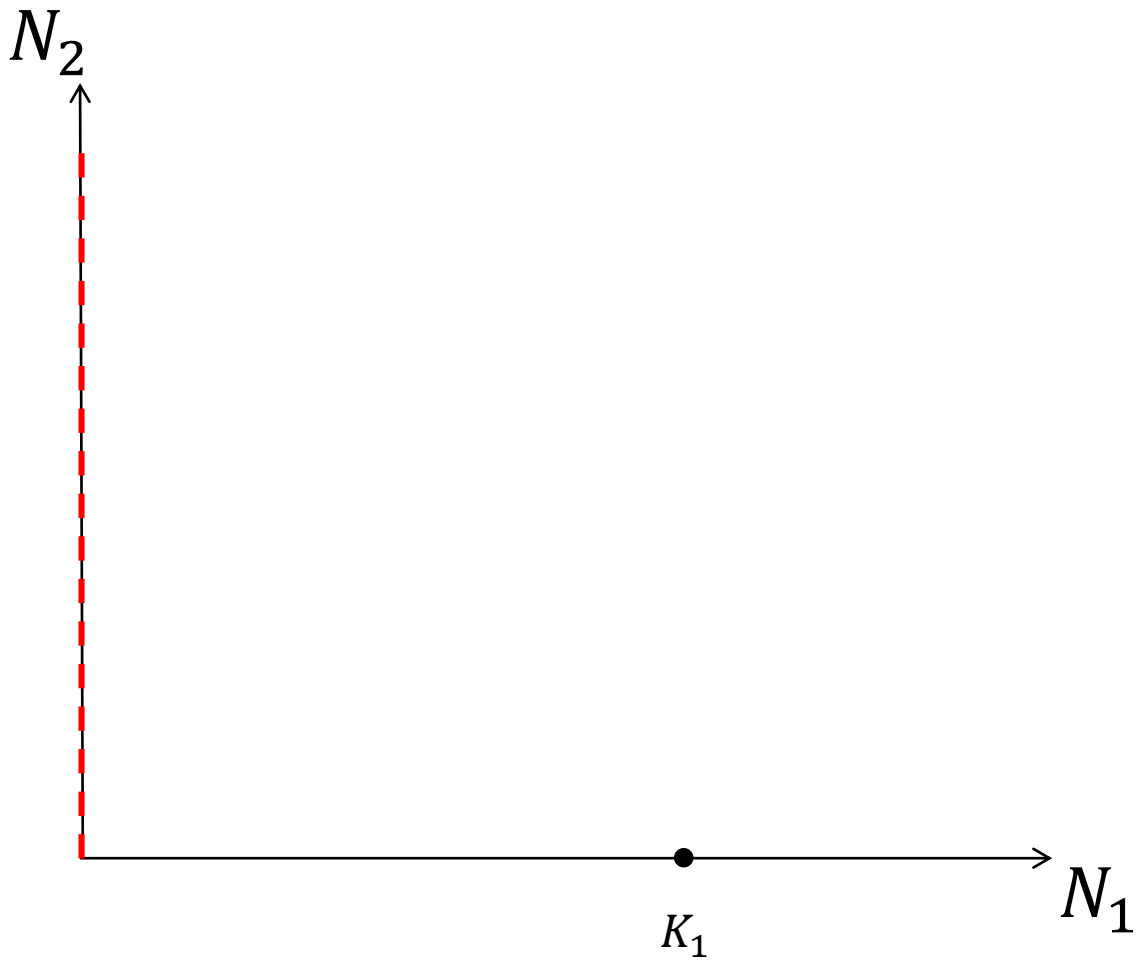
- Czy zające i owce mogą koegzystować?
- Czy jeden z gatunków zdominuje drugiego?

- Nullclines:

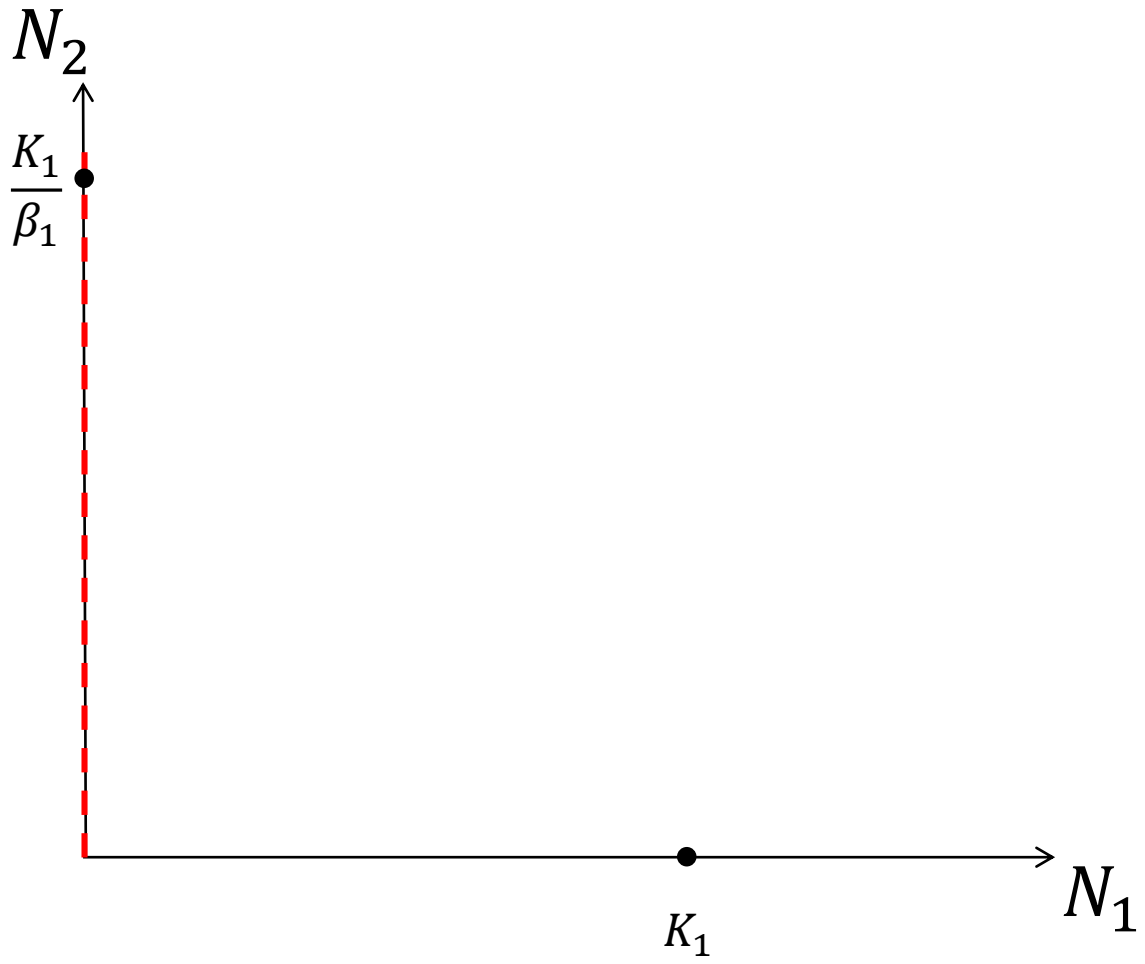
- $N_1 = 0$ lub $K_1 - N_1 - \beta_1 N_2 = 0$

Nullclines

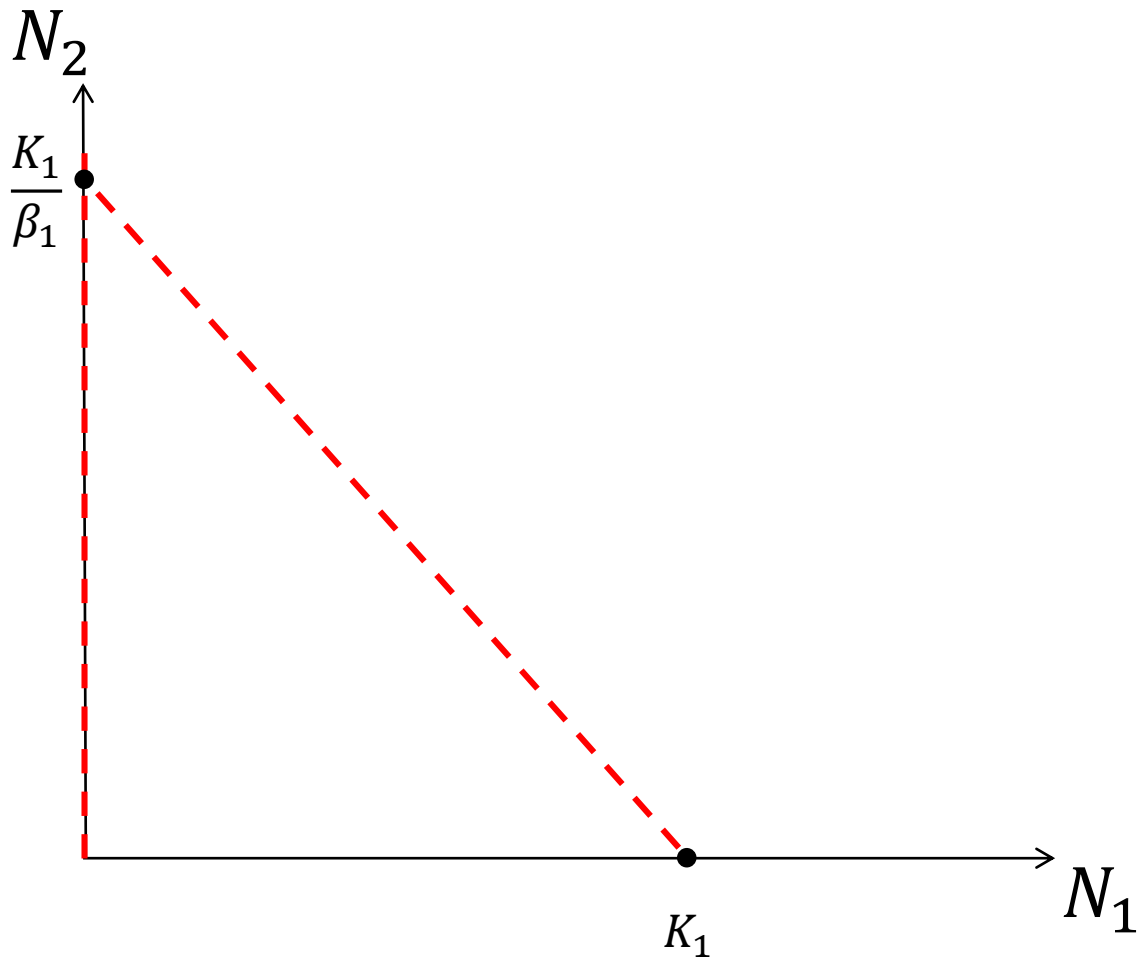
Nullclines



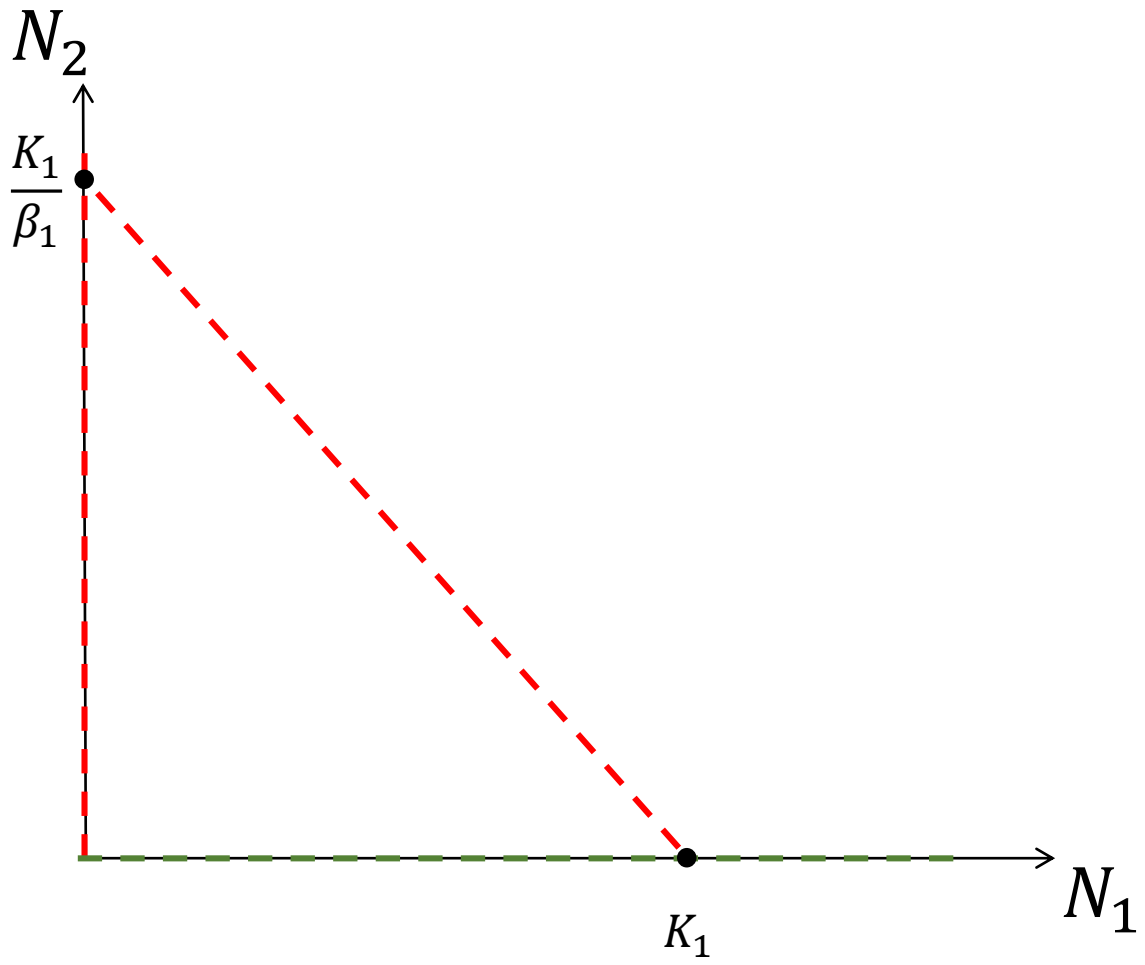
Nullclines



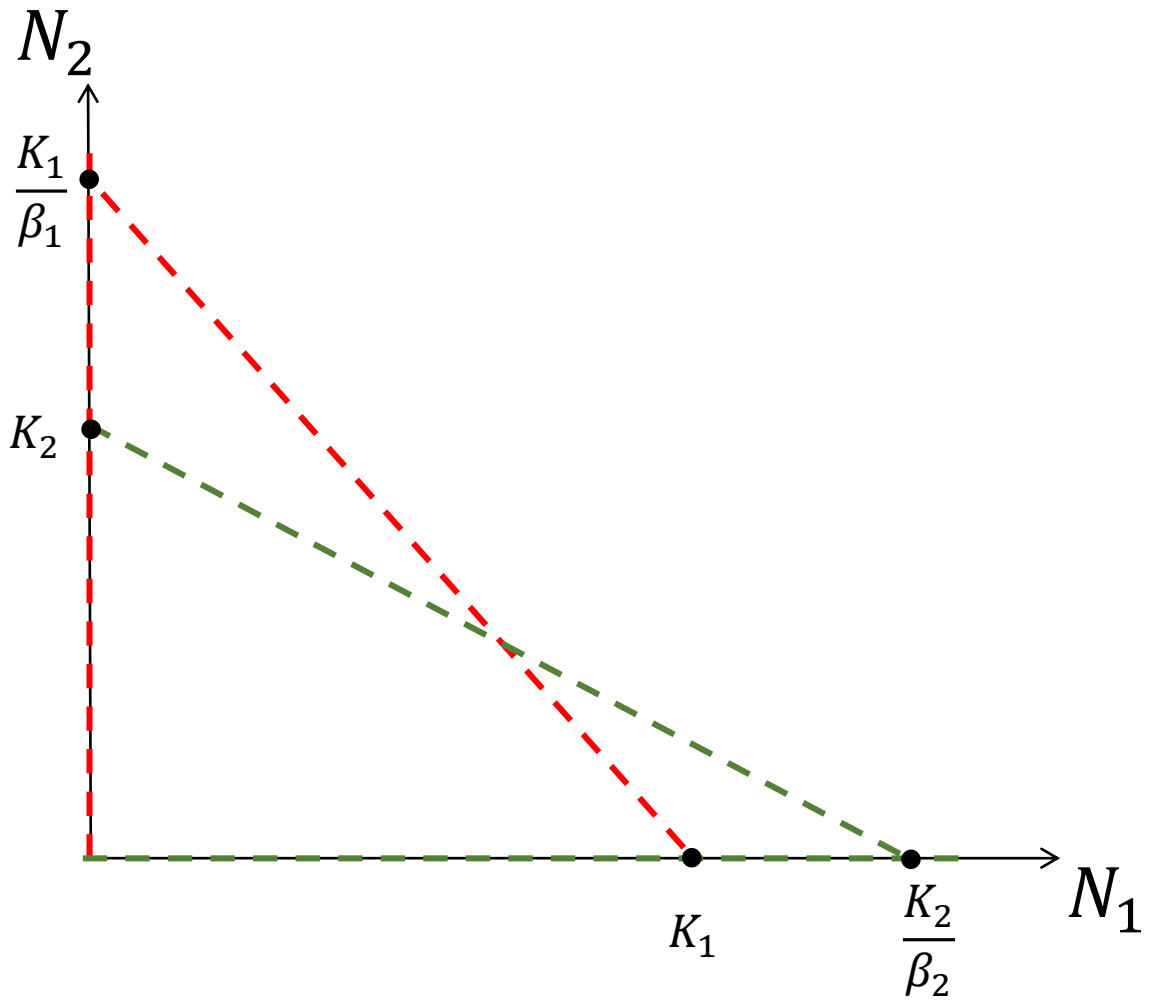
Nullclines



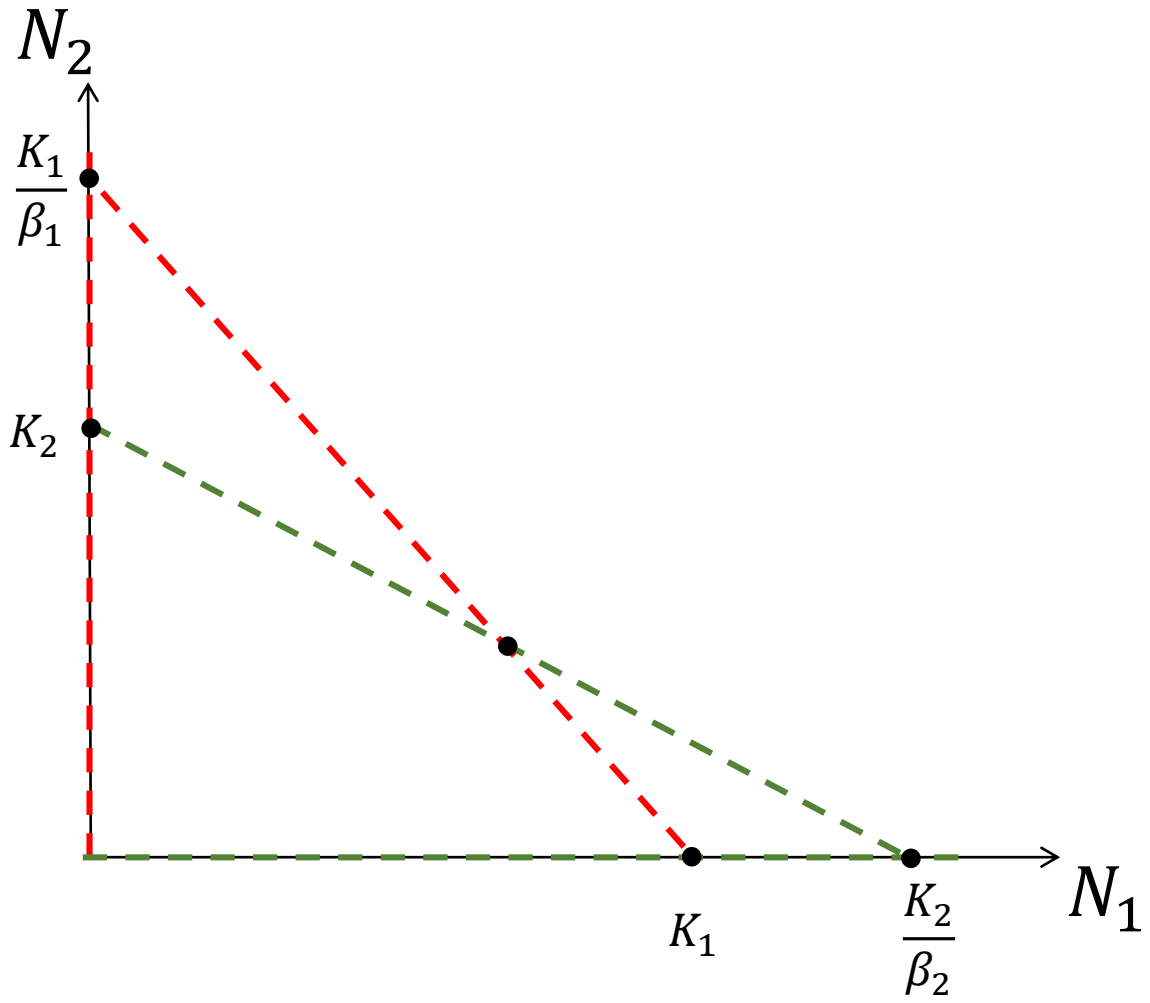
Nullclines



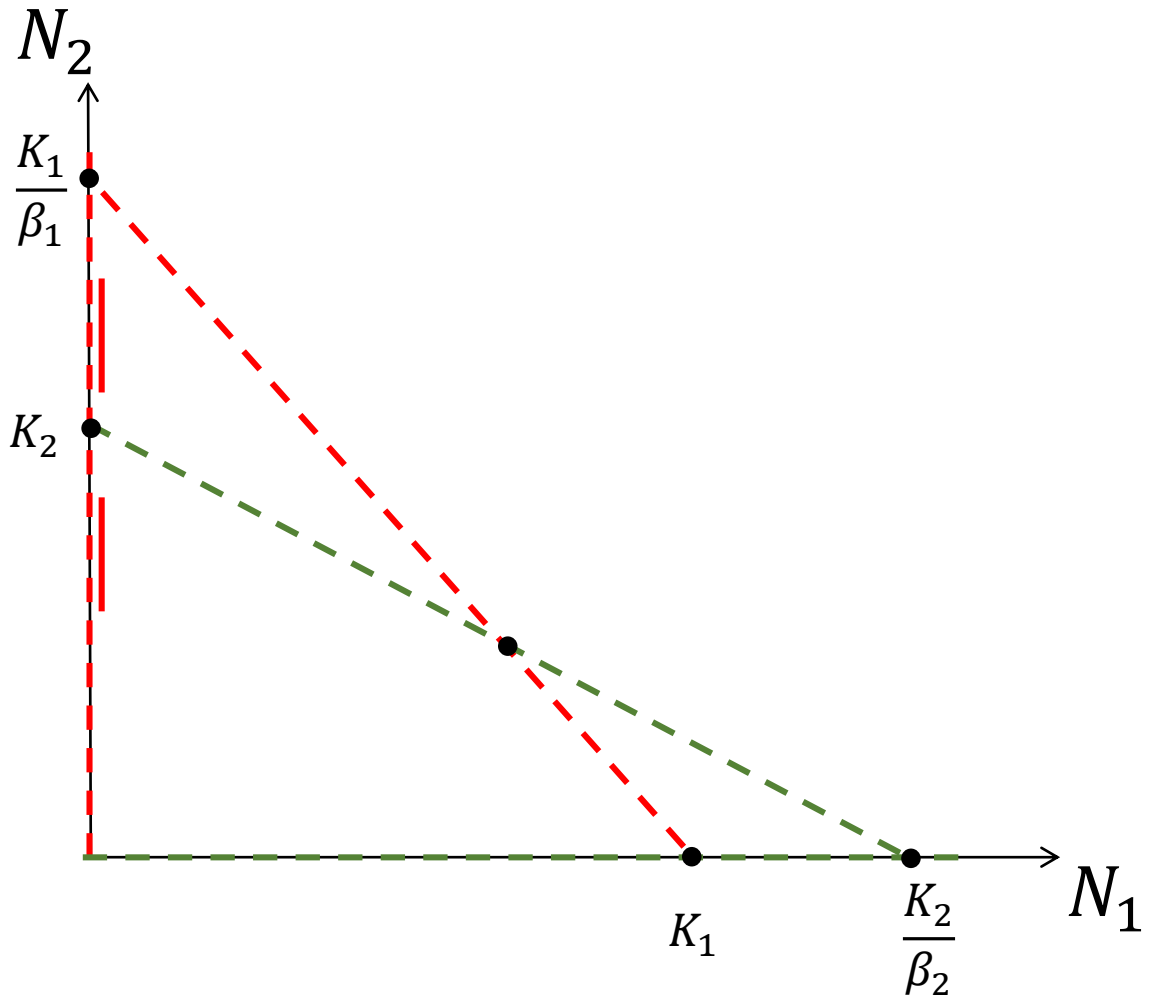
Nullclines



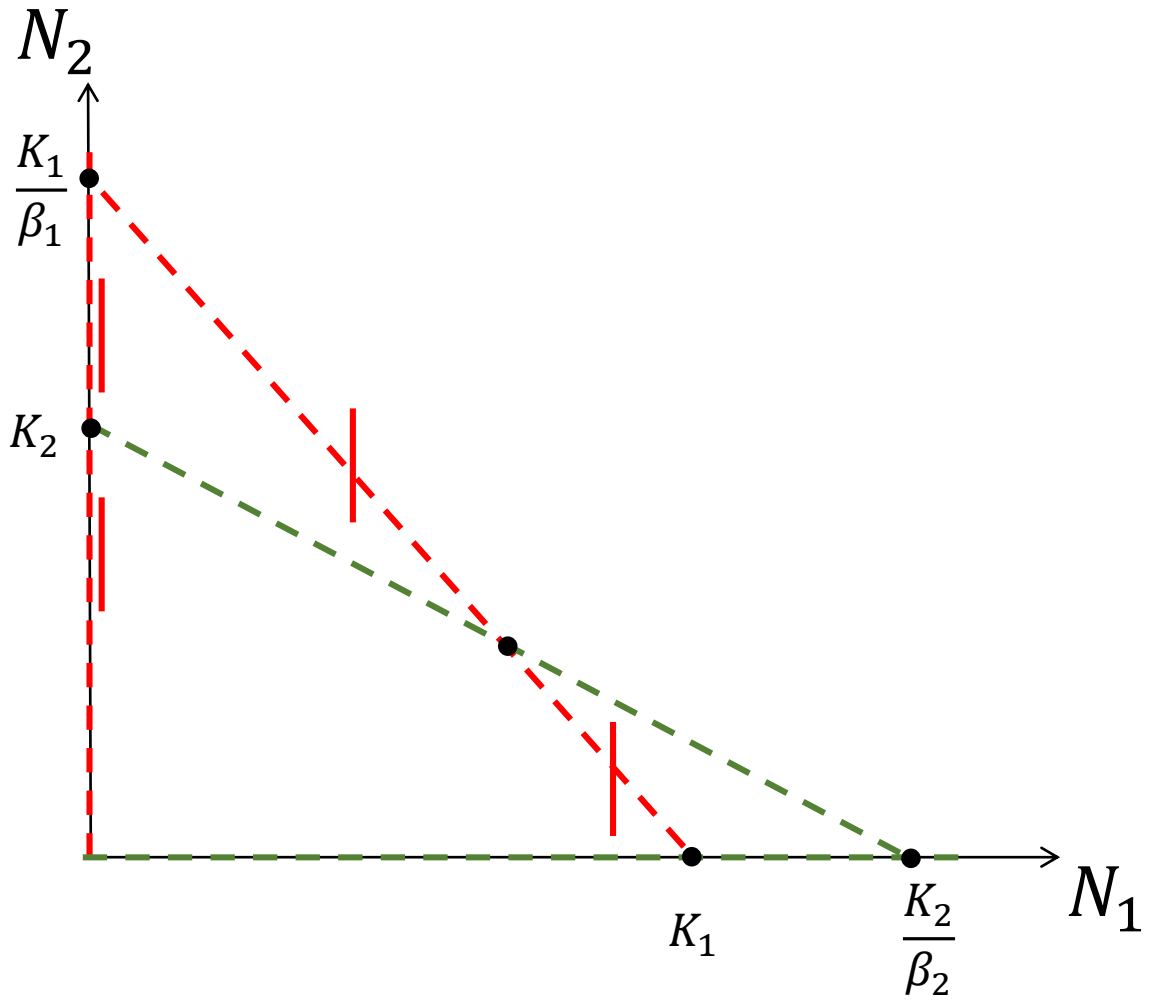
Nullclines



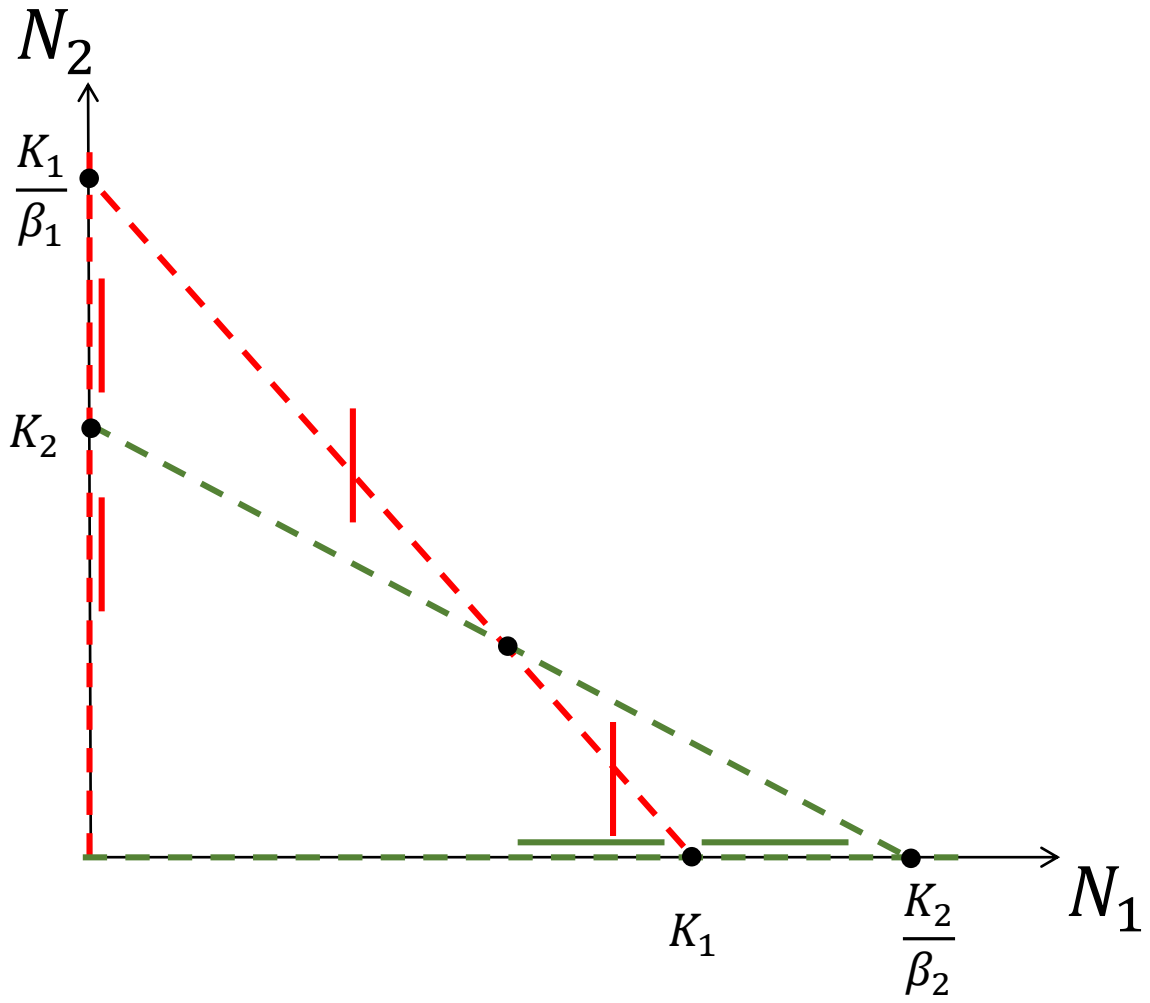
Nullclines



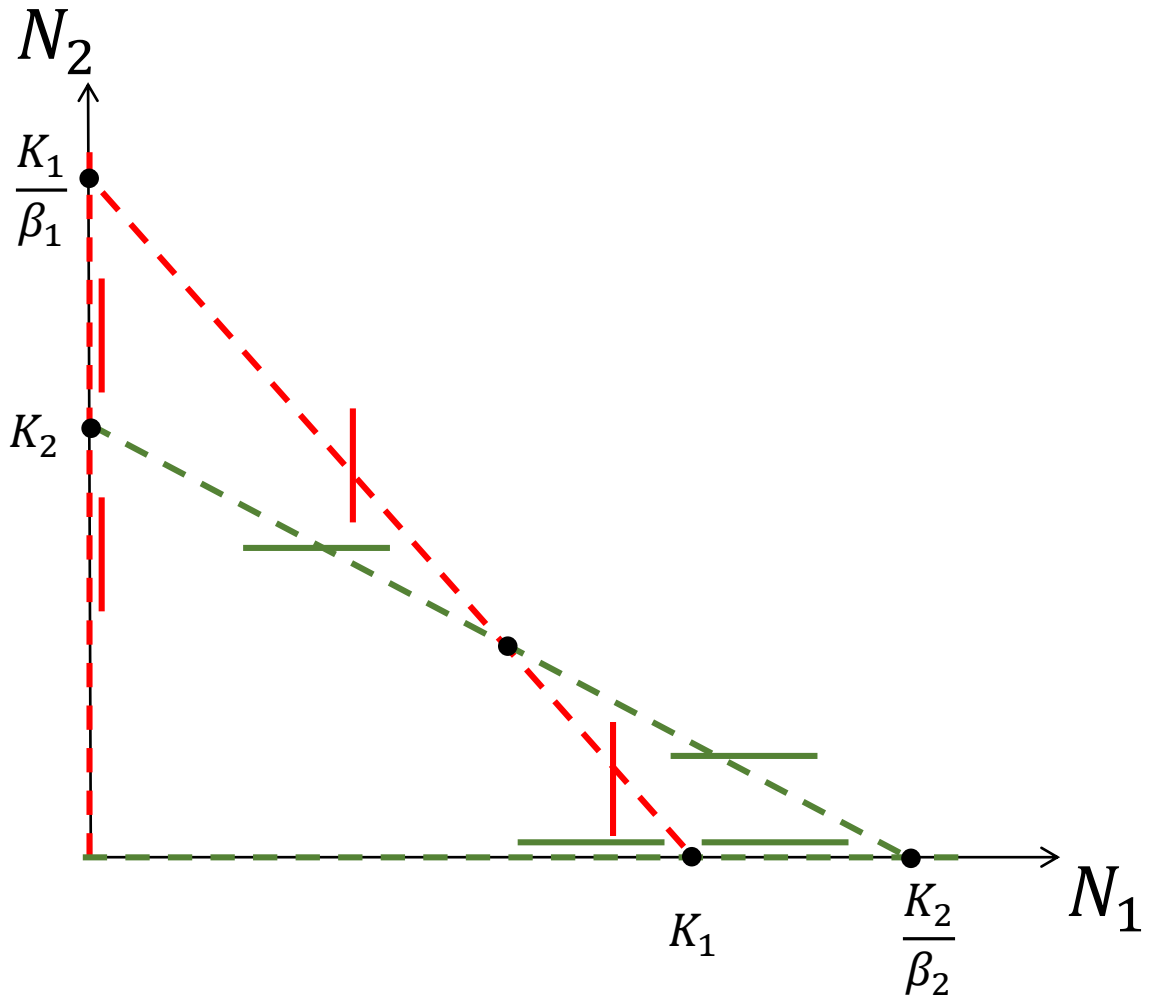
Nullclines



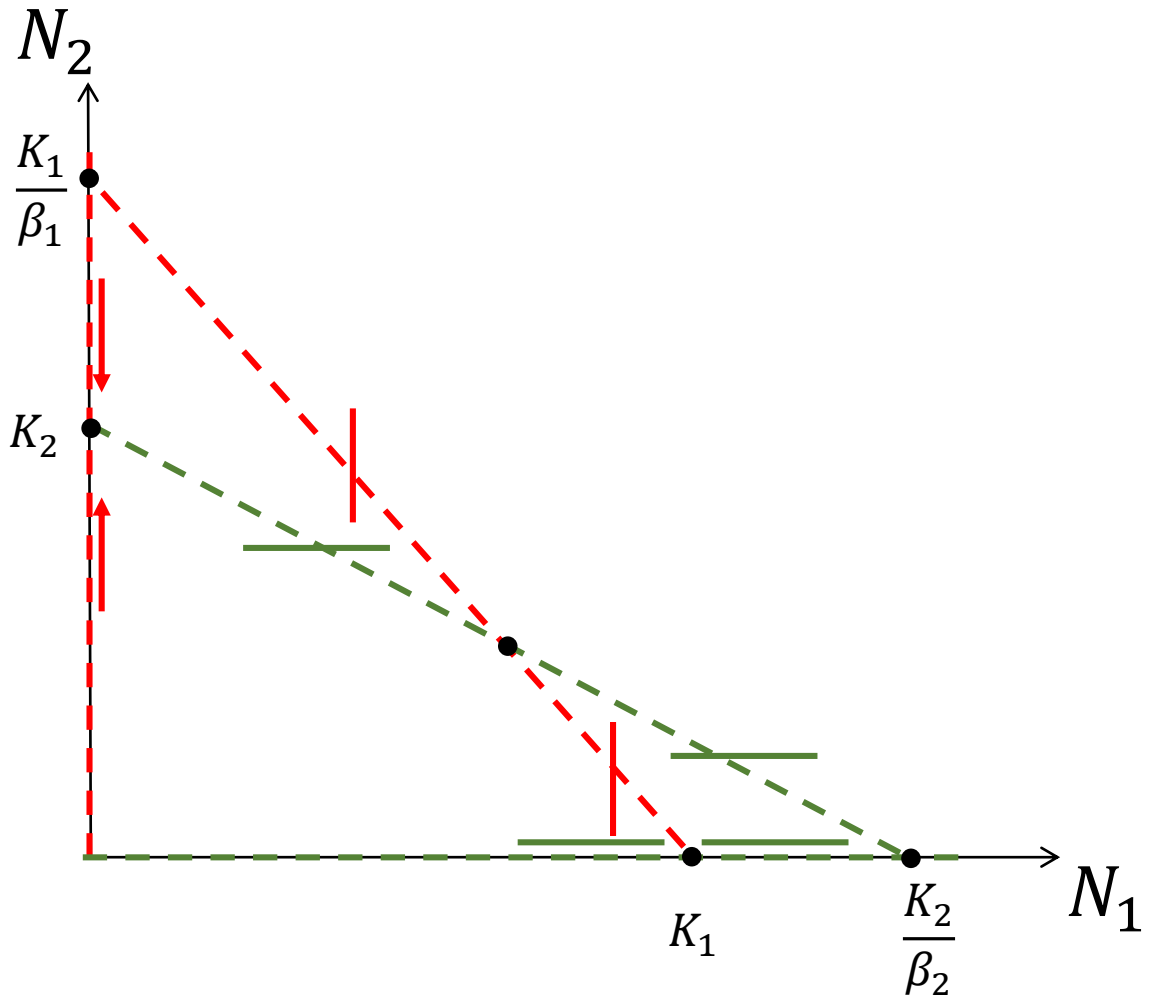
Nullclines



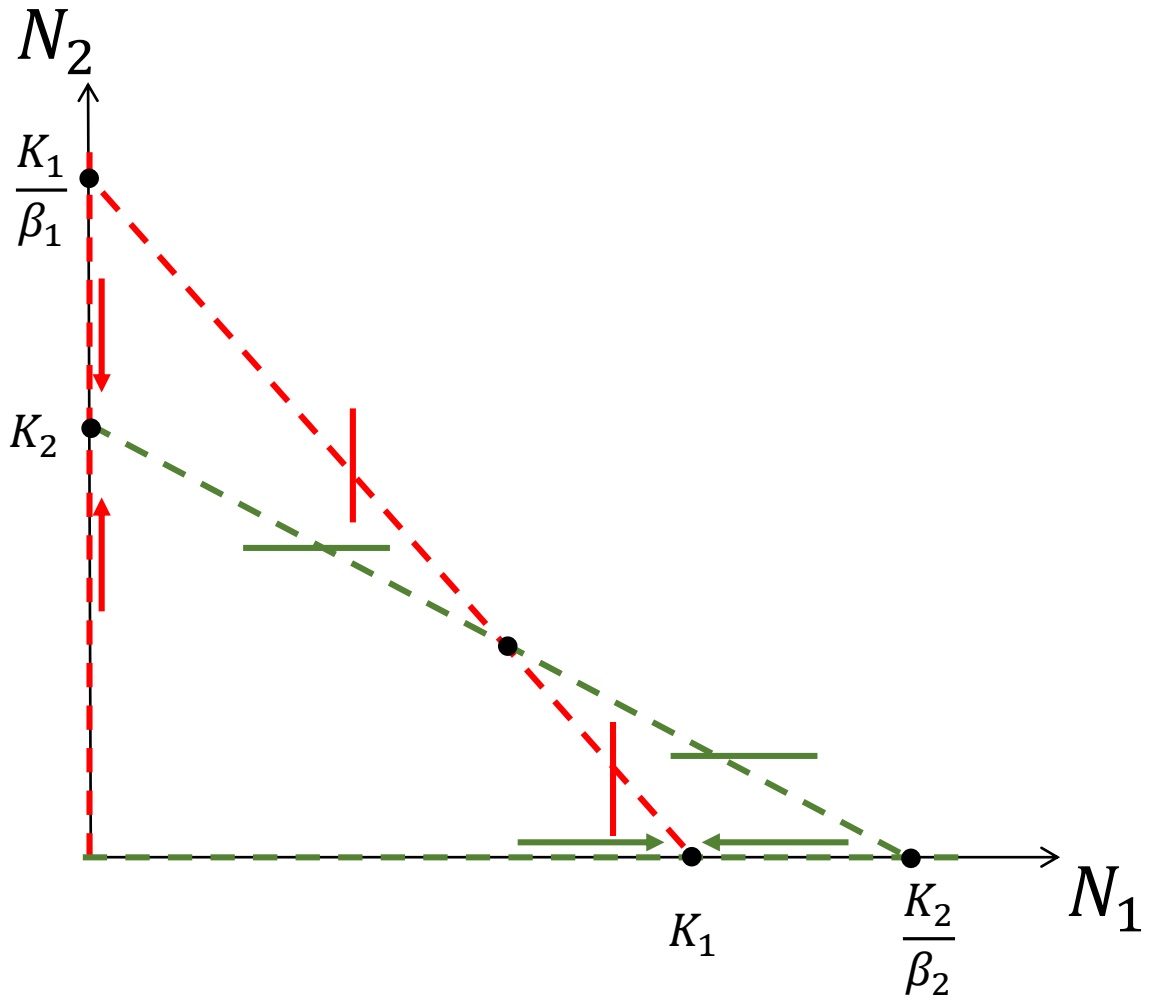
Nullclines



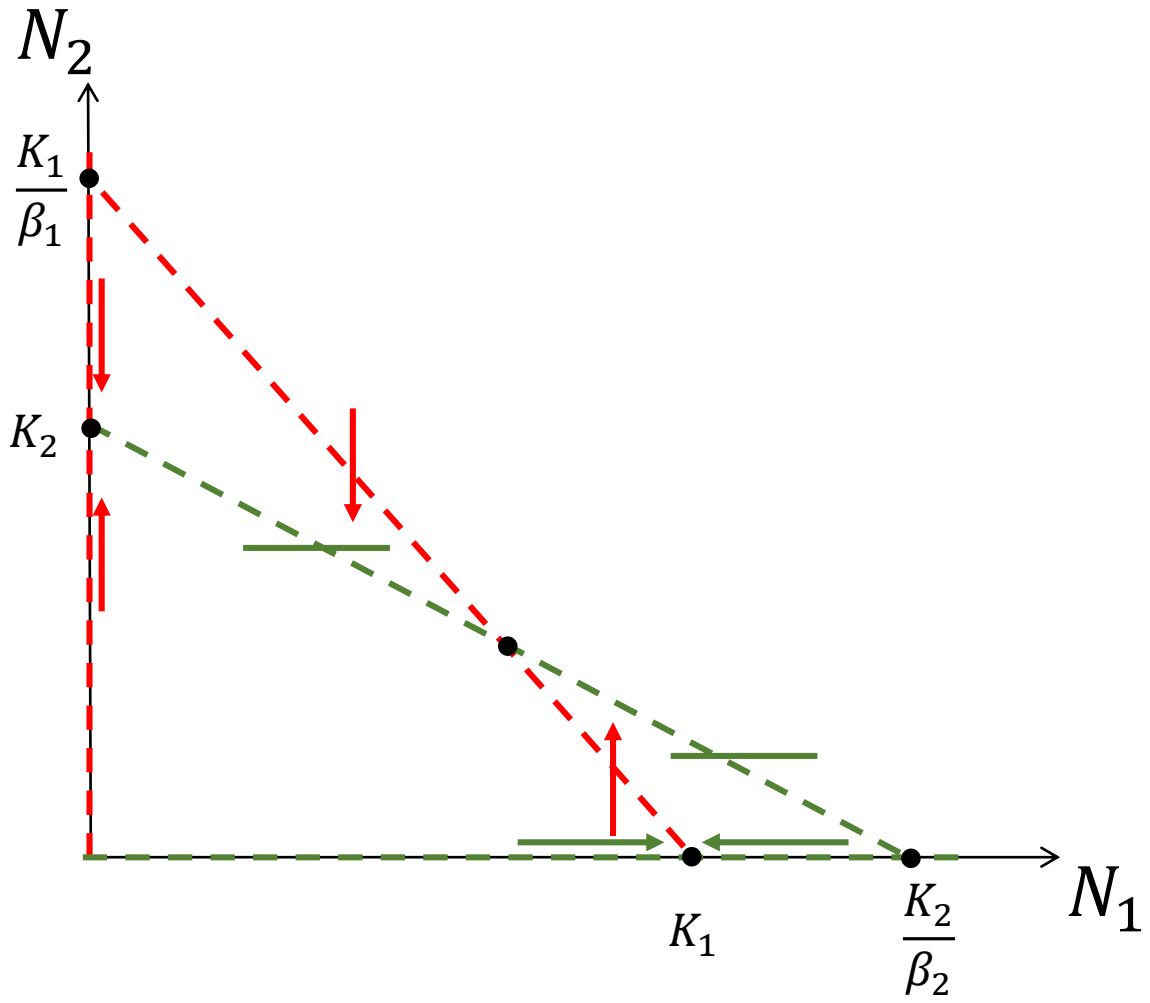
Nullclines



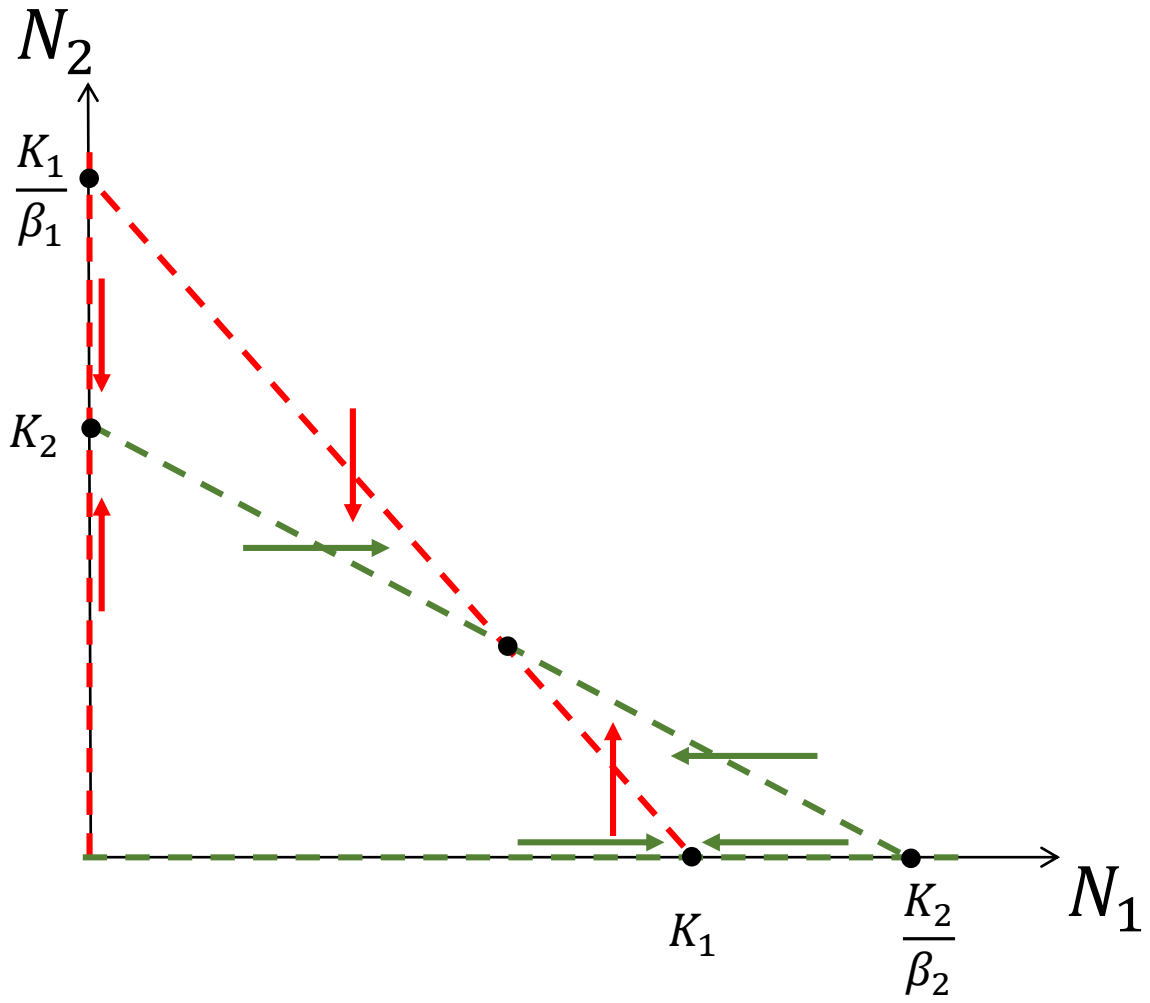
Nullclines



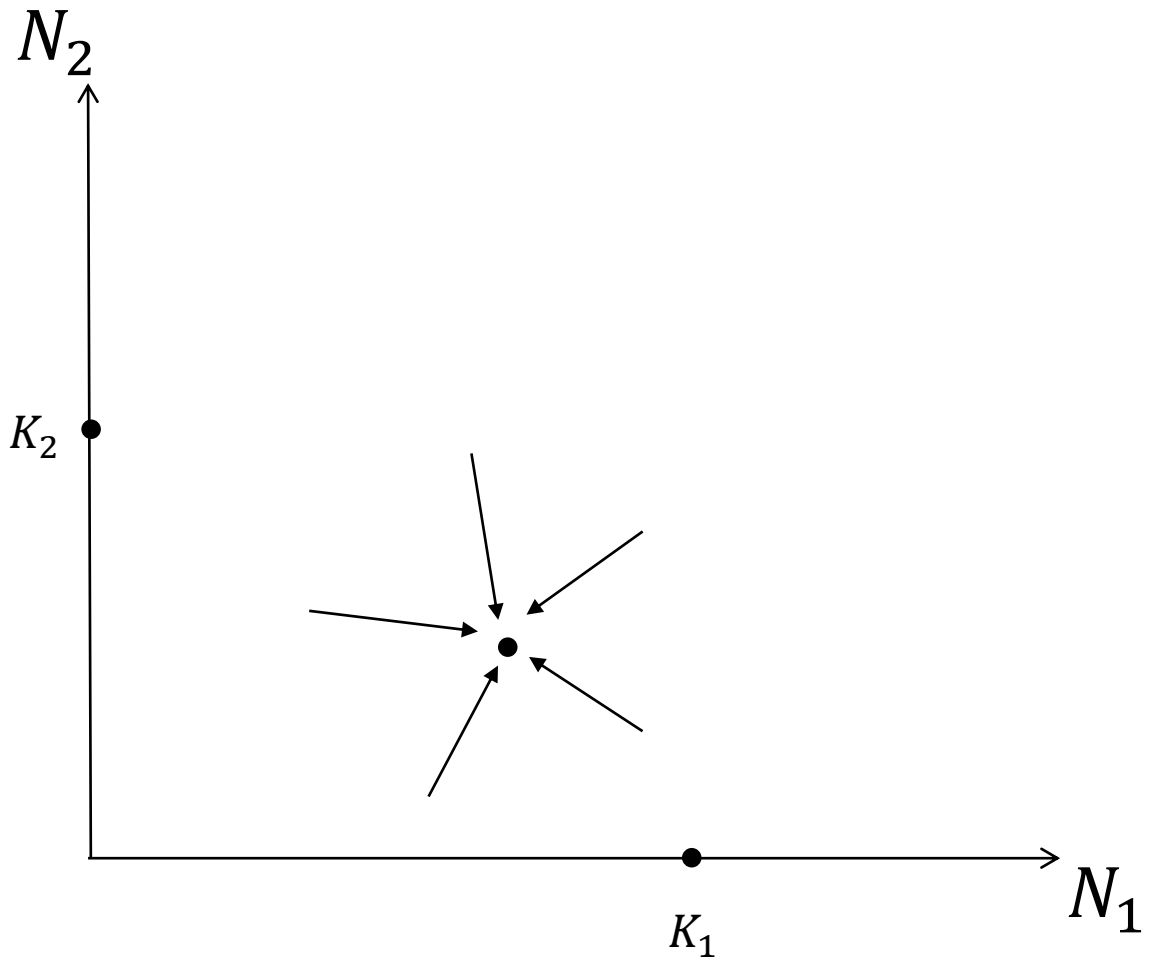
Nullclines



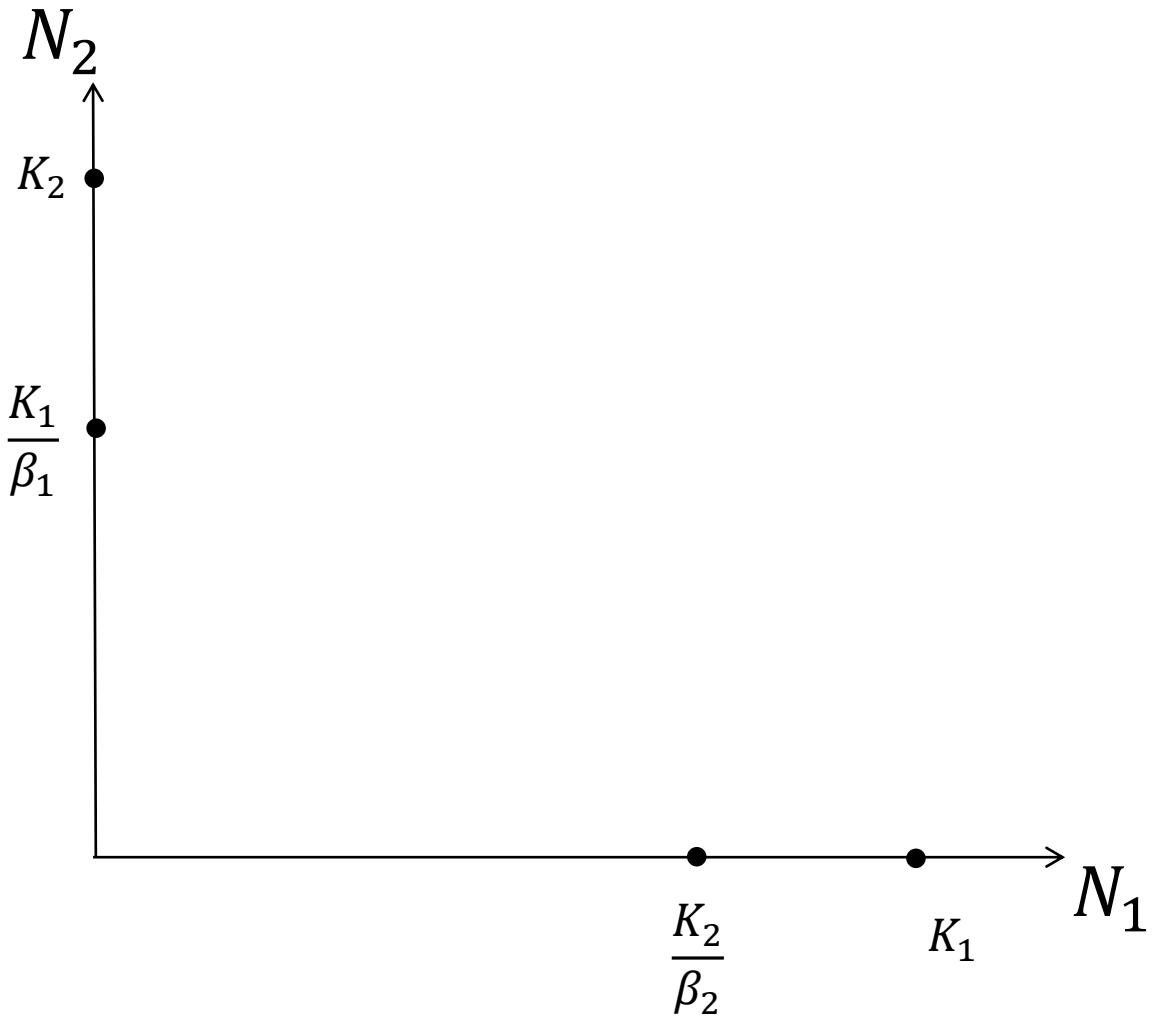
Nullclines



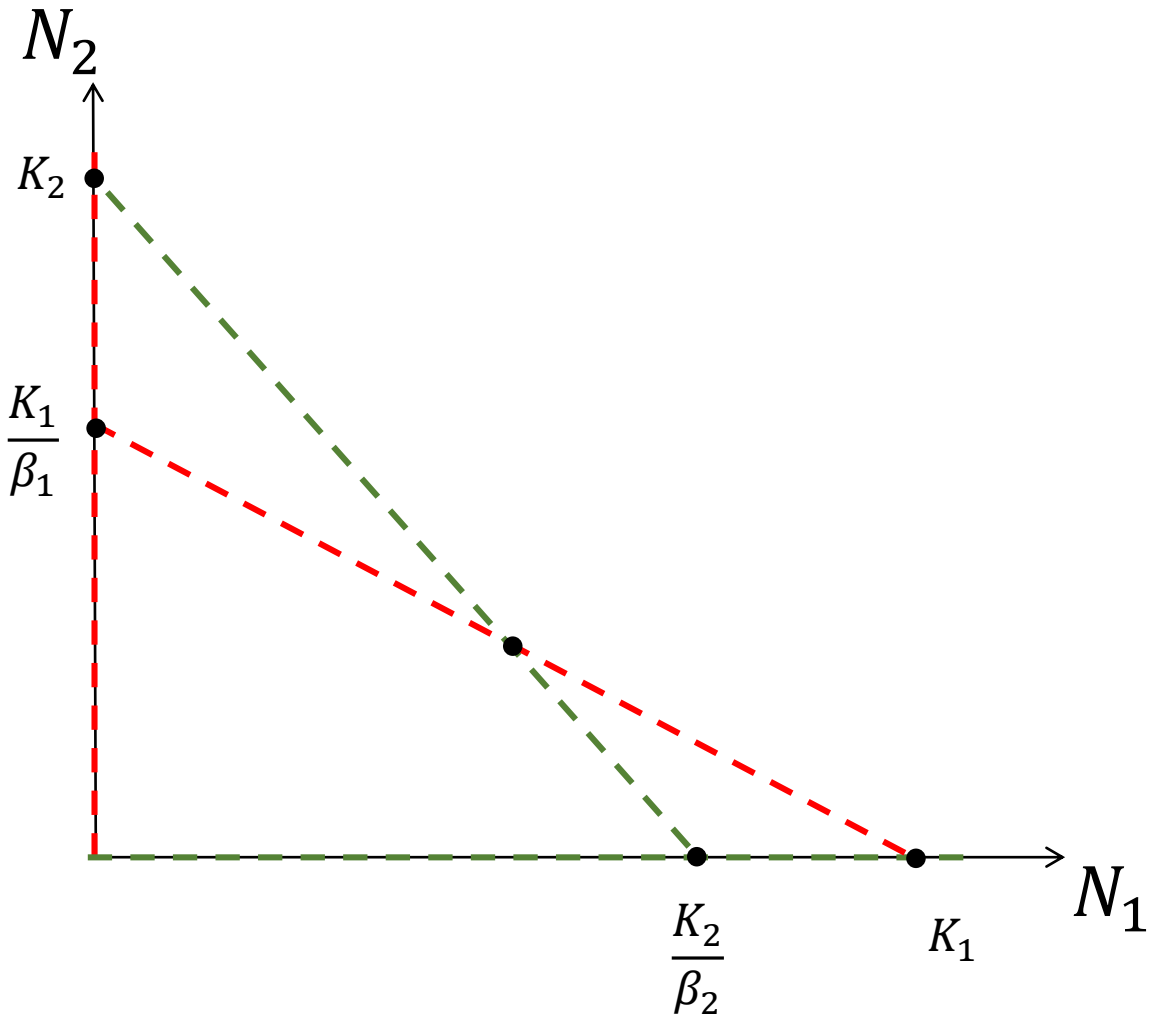
Stabilność – gatunki koegzystują



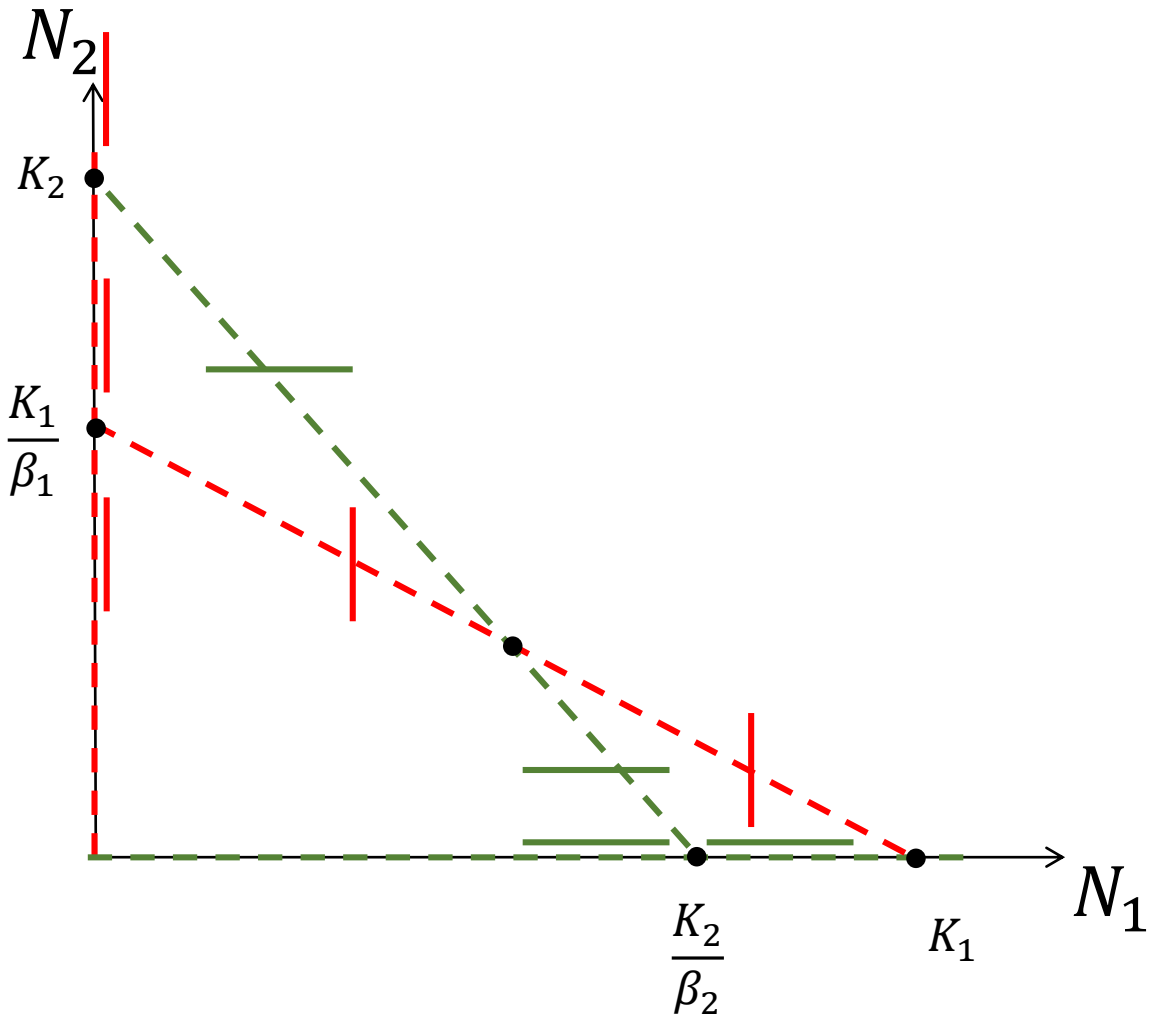
Nullclines: przypadek duże β_i



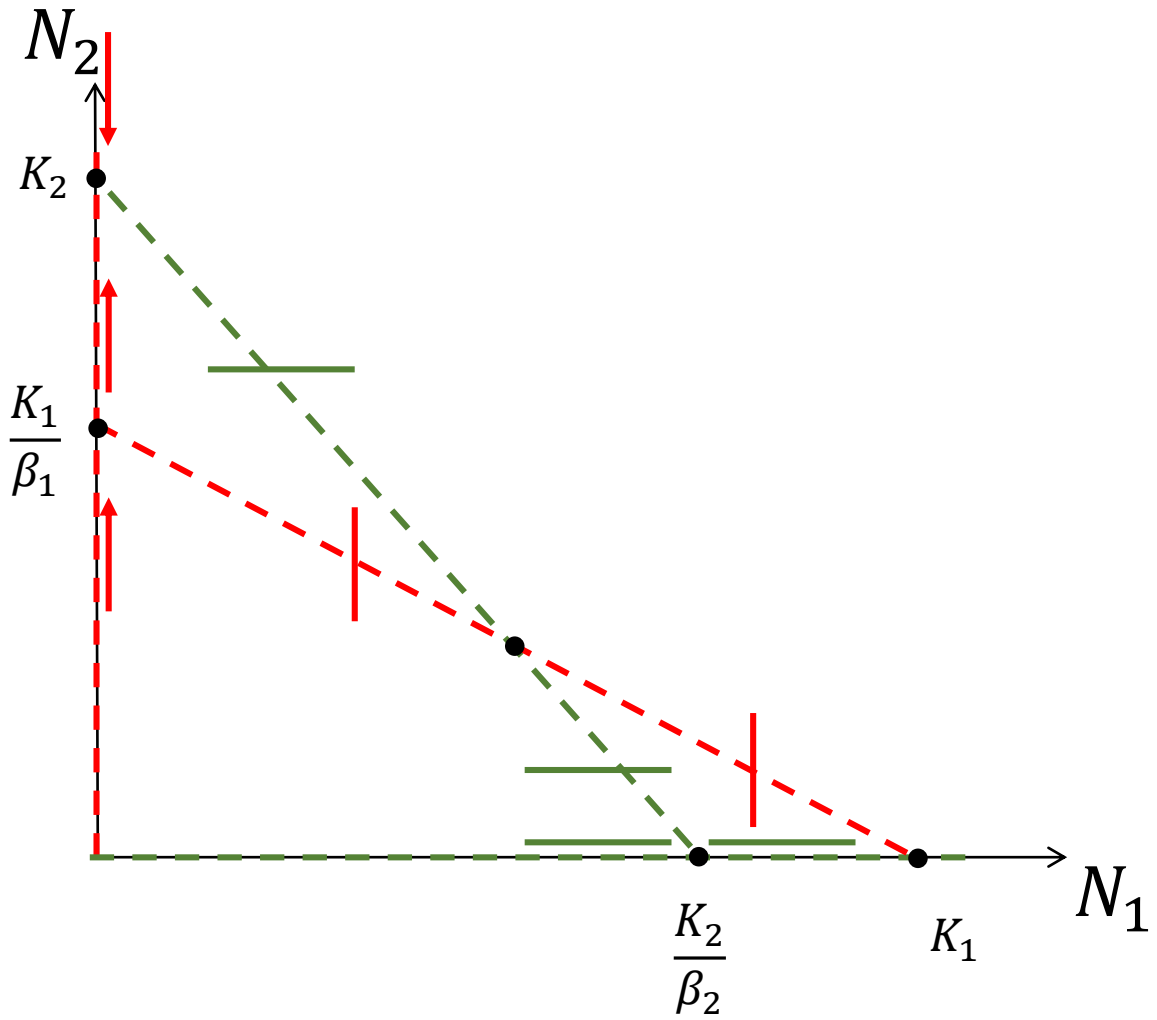
Nullclines: przypadek duże β_i



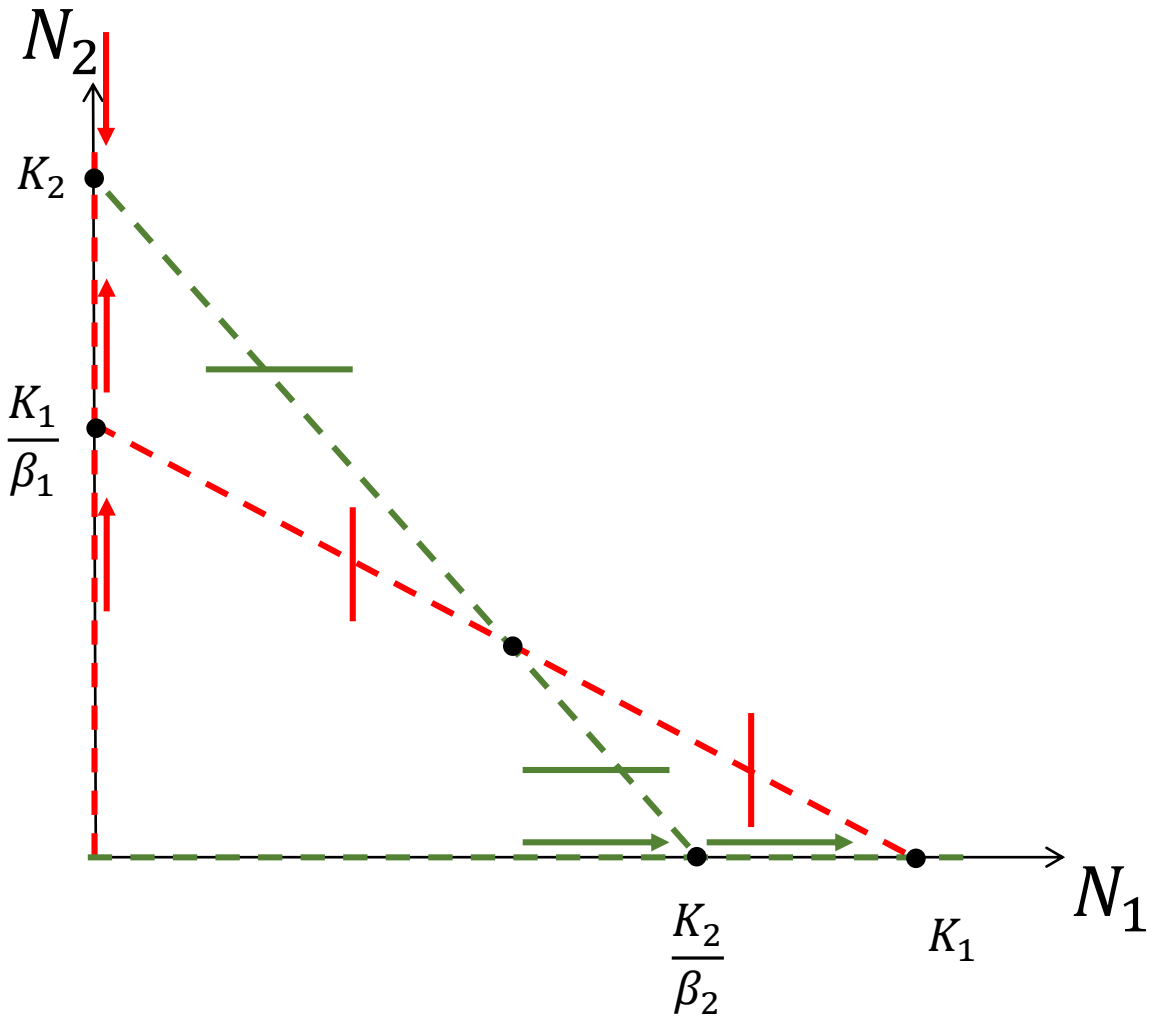
Nullclines: przypadek duże β_i



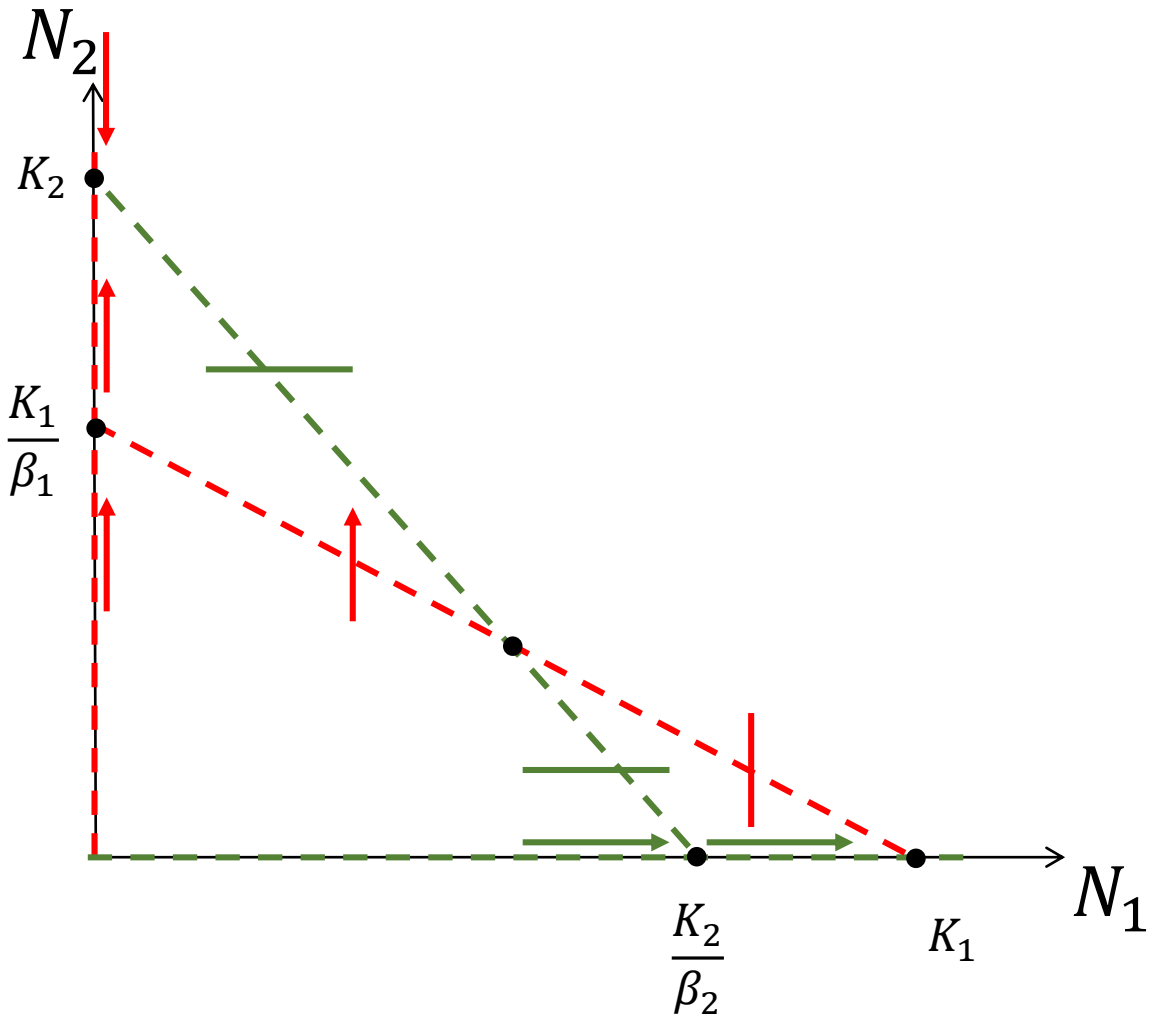
Nullclines: przypadek duże β_i



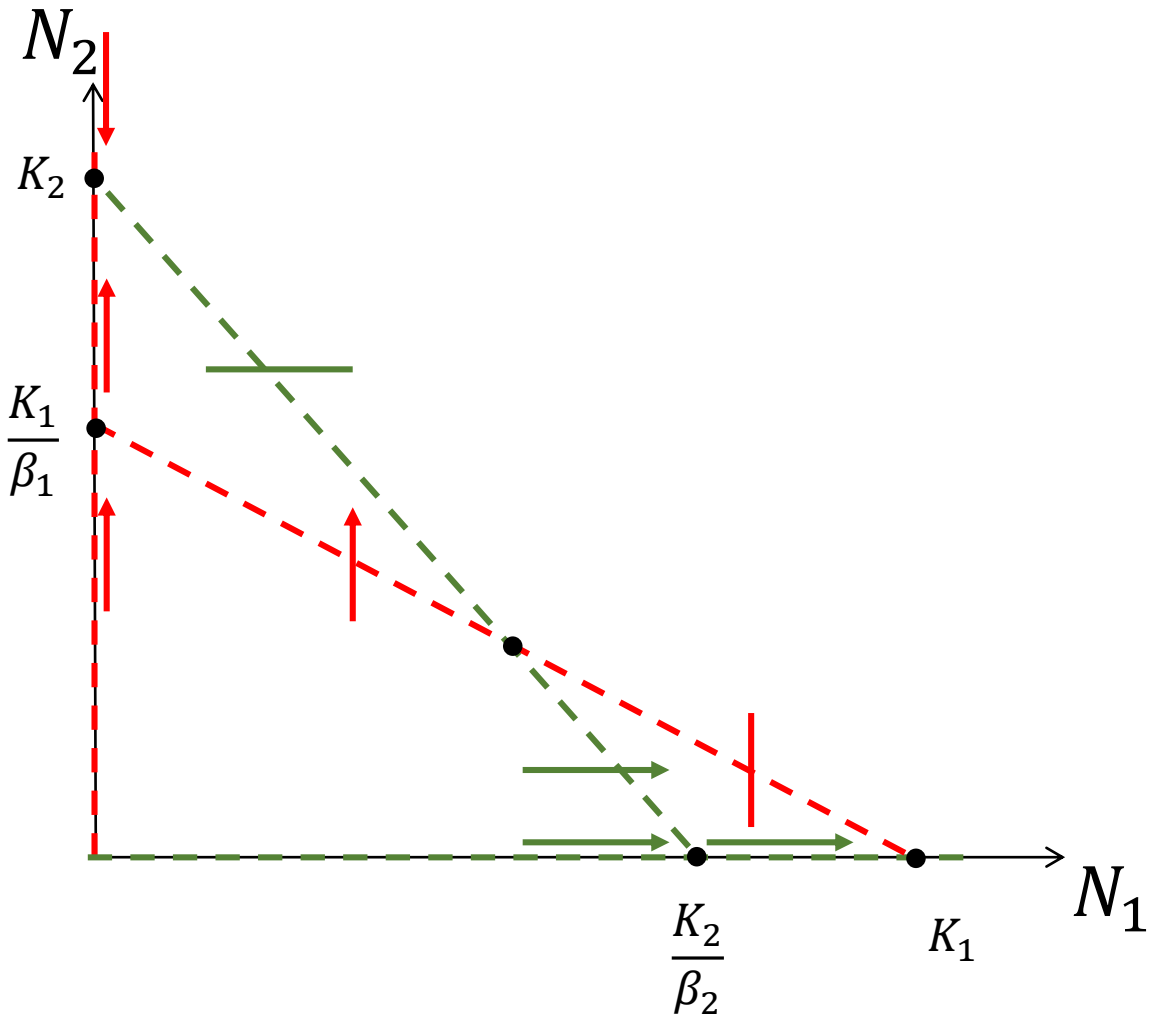
Nullclines: przypadek duże β_i



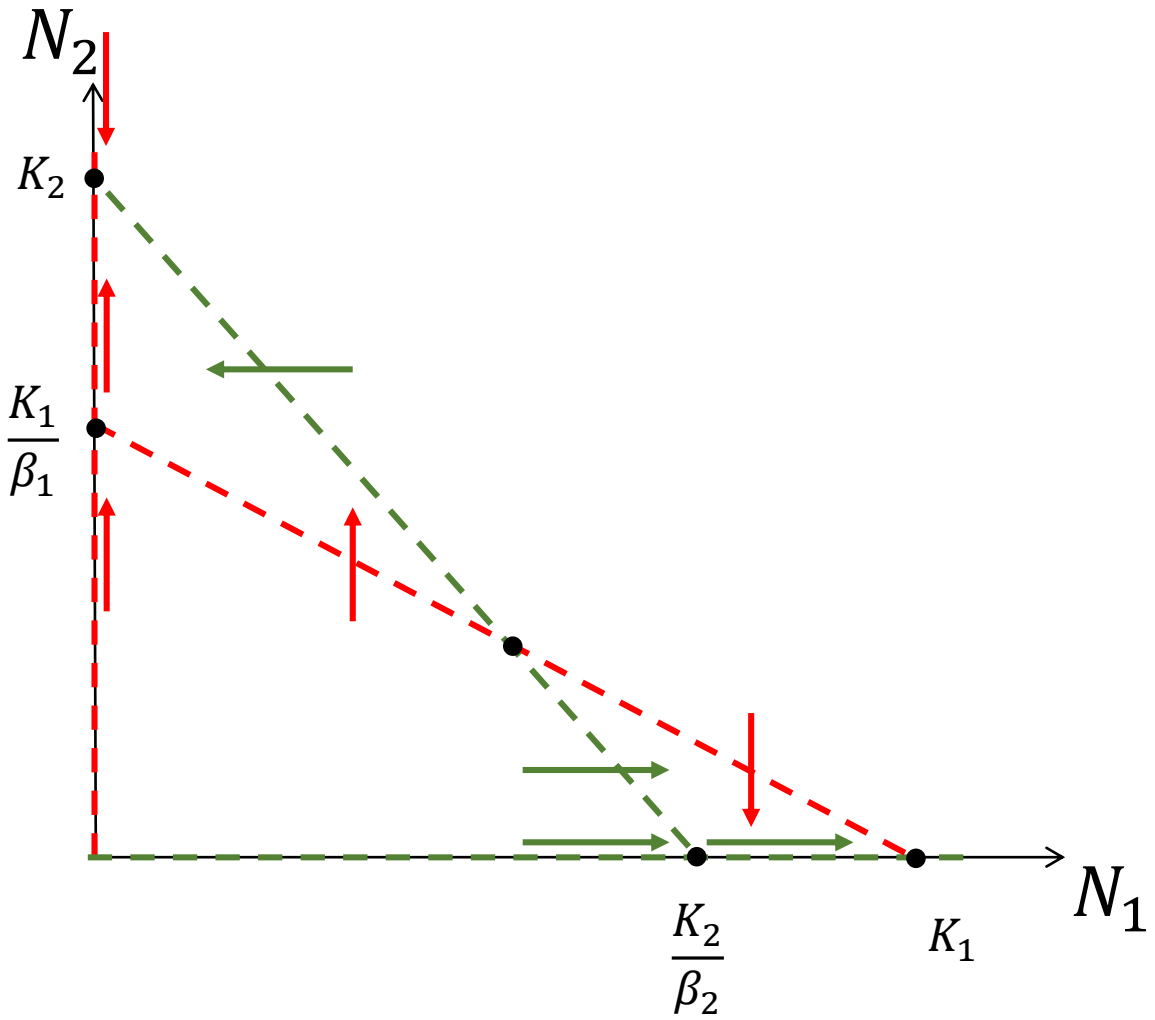
Nullclines: przypadek duże β_i



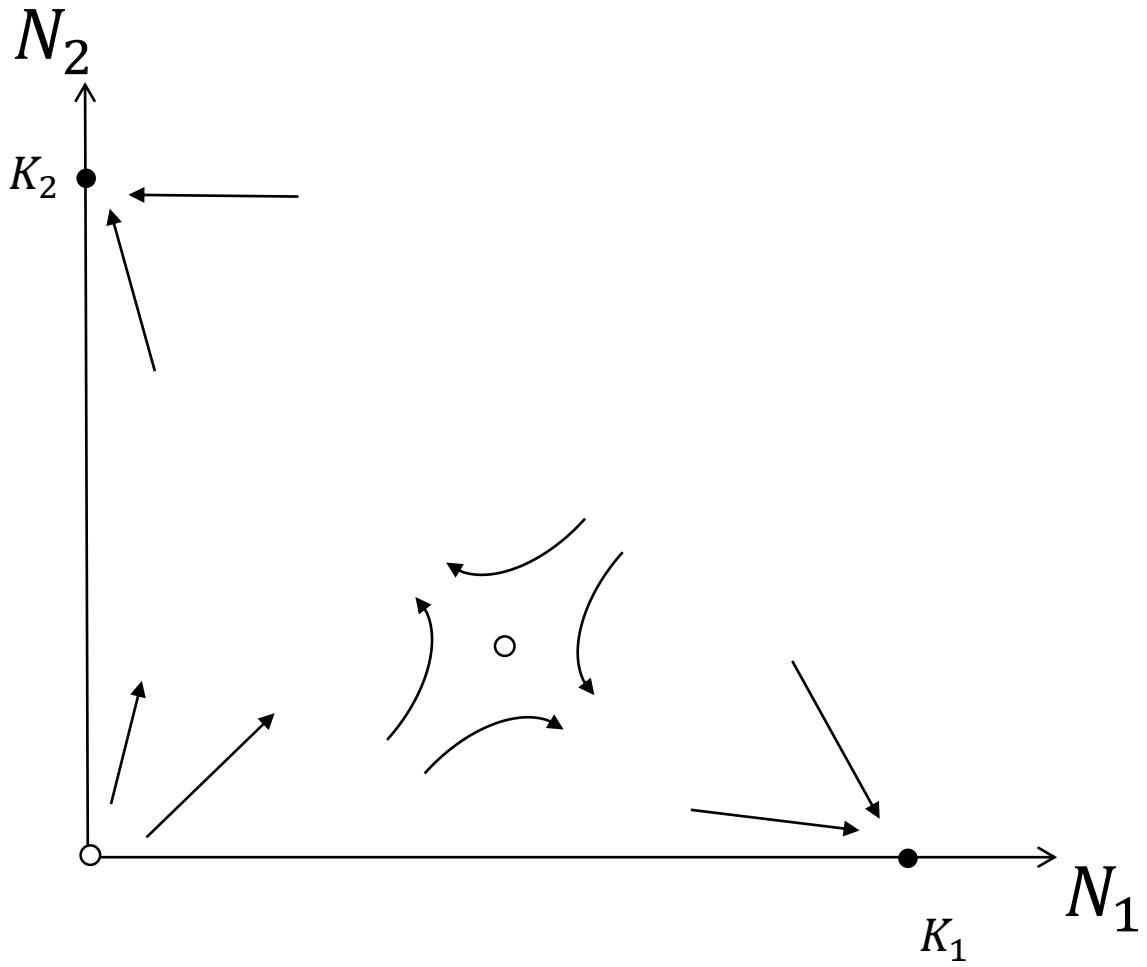
Nullclines: przypadek duże β_i

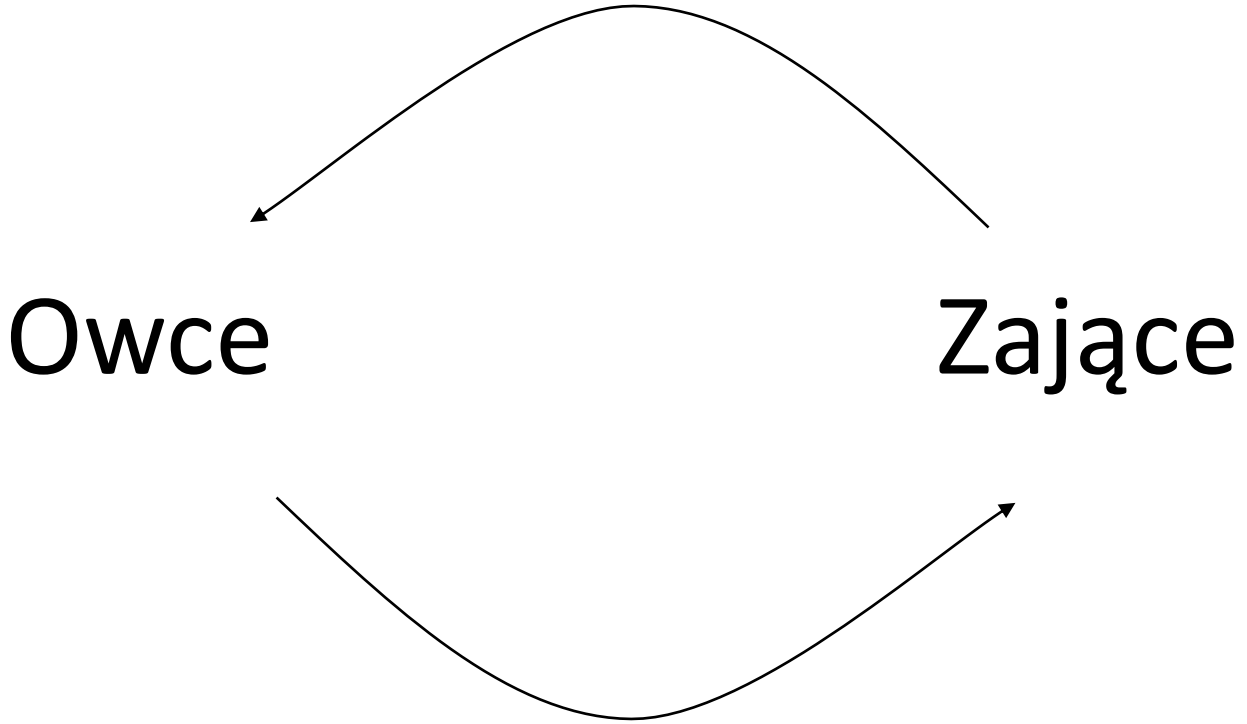


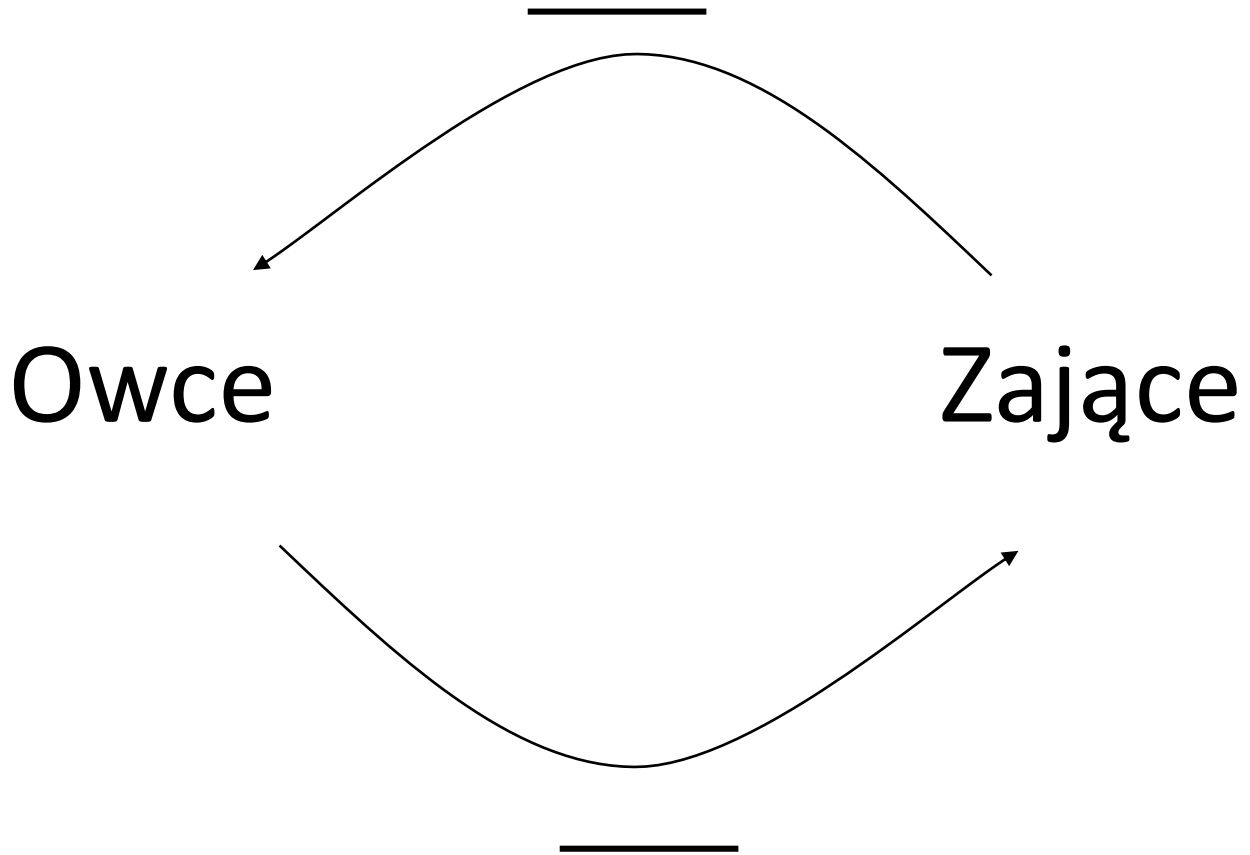
Nullclines: przypadek duże β_i



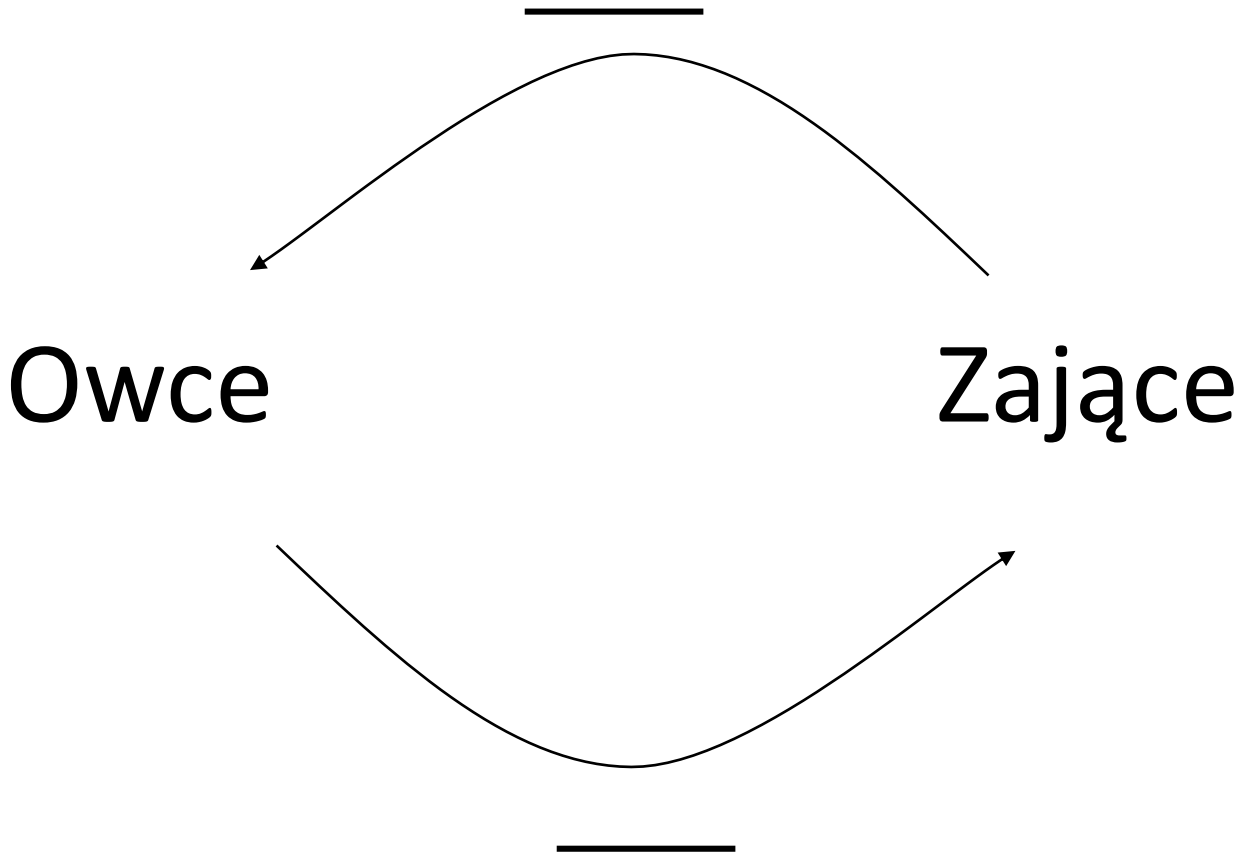
Stabilność – gatunki nie koegzystują




Owce
Zające


Owce
Zające

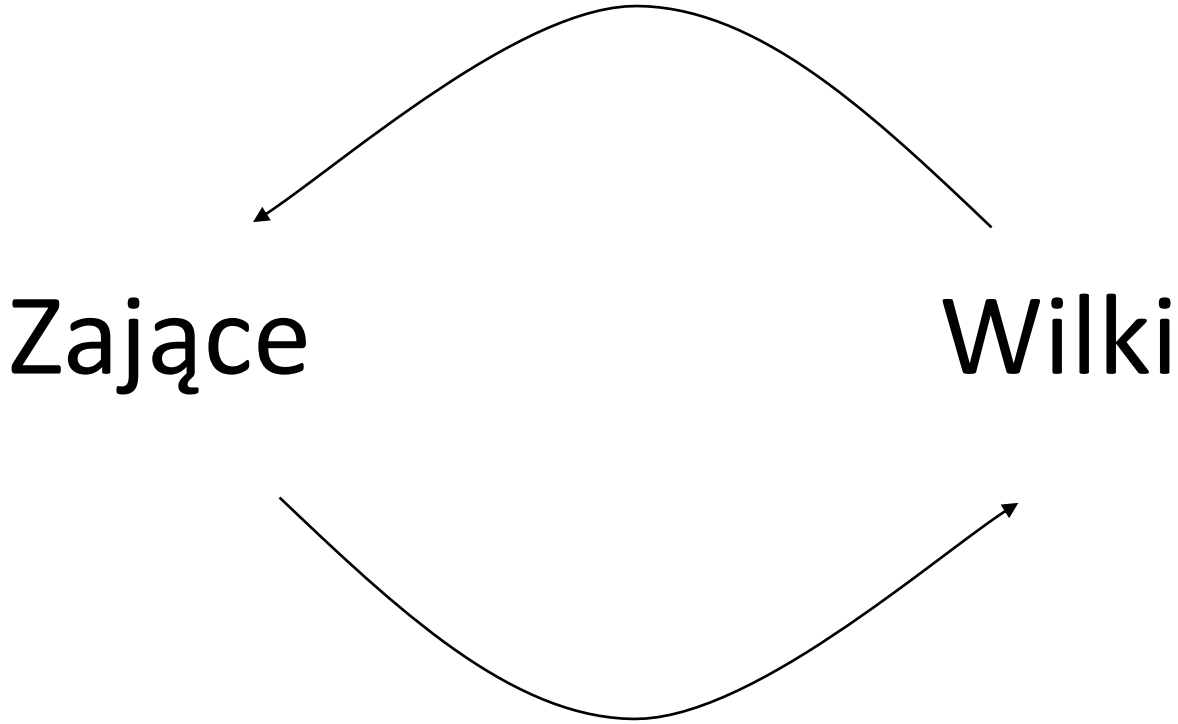
Positive feedback



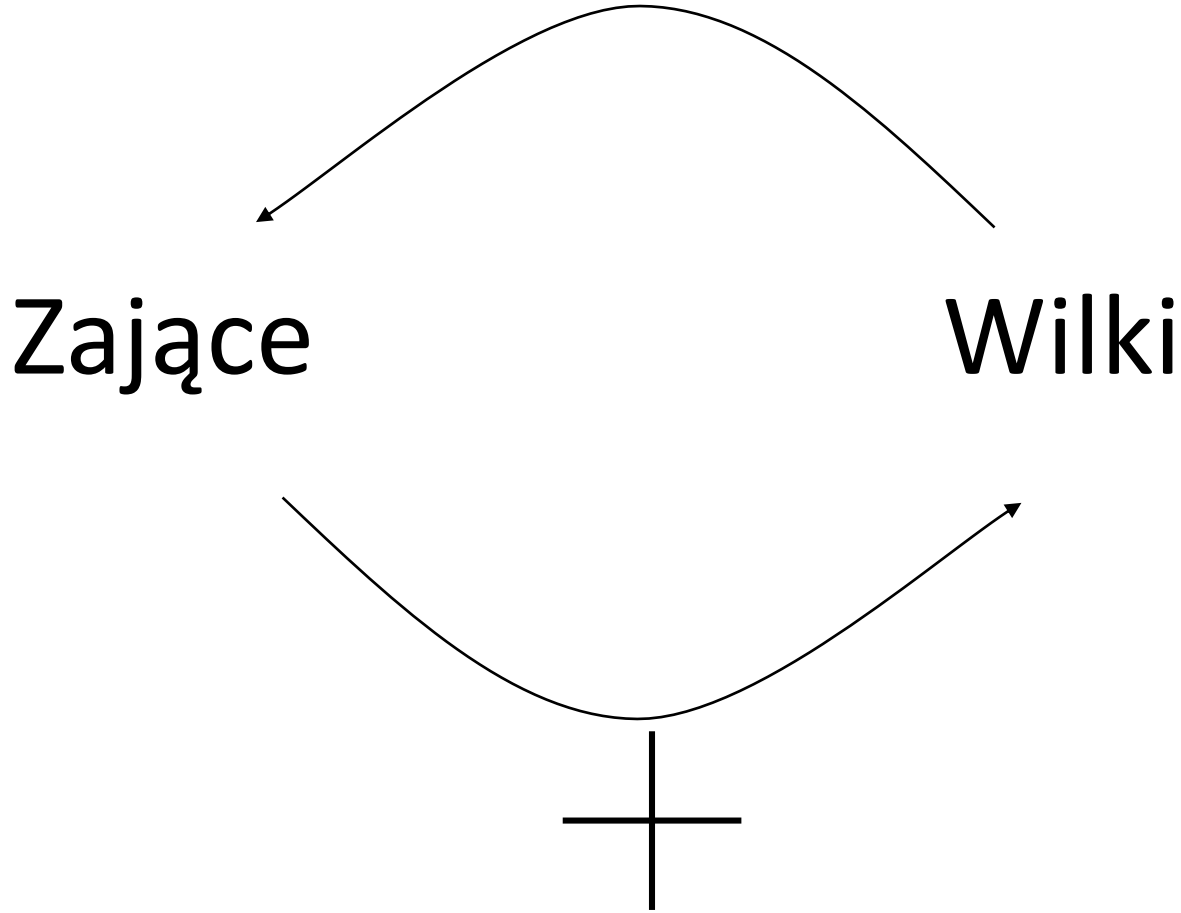
Multistability

- Sprężenie zwrotne dodatnie może prowadzić to kliku punktów stacjonarnych w systemie

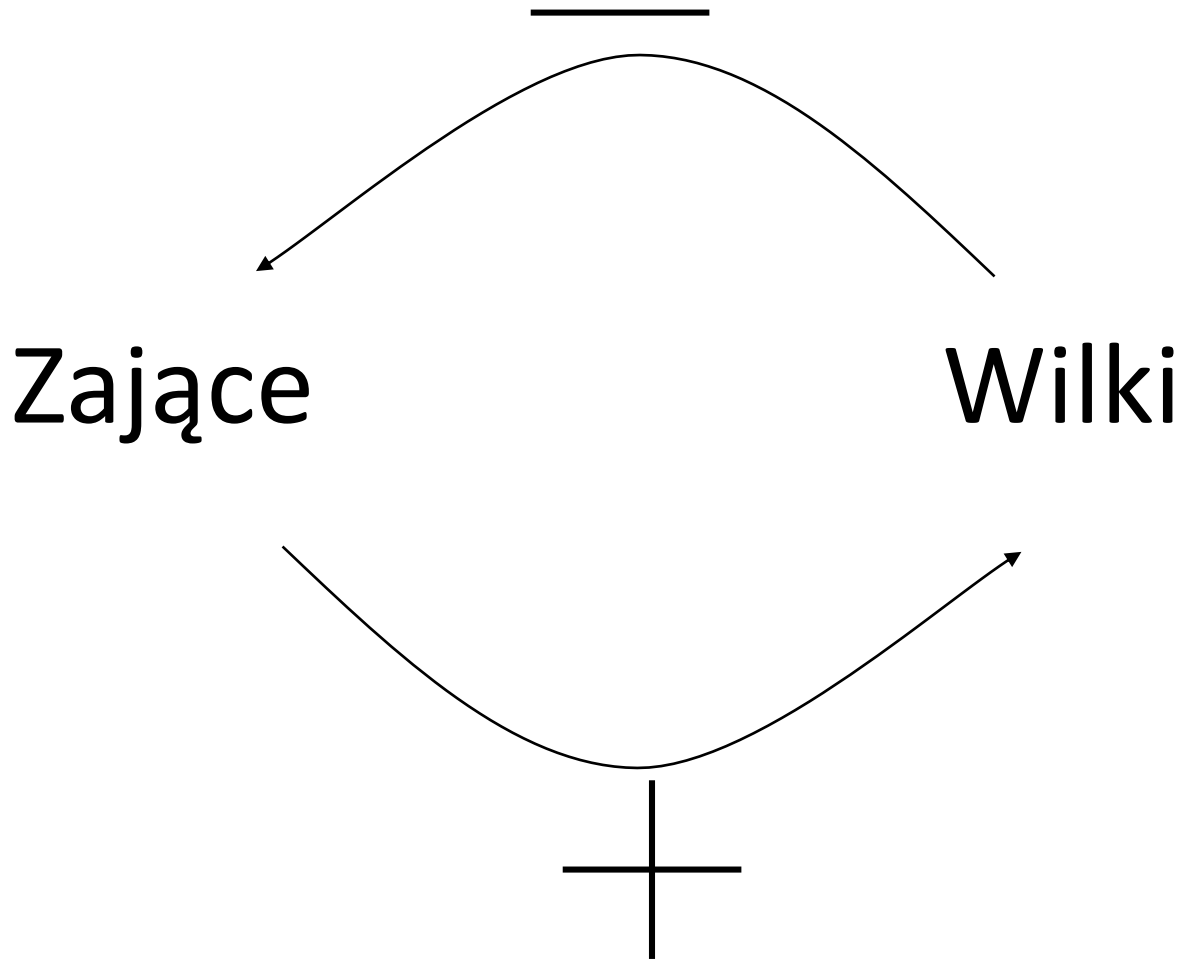
Drapieżnik-Ofiara



Drapieżnik-Ofiara



Negative Feedback



Populacje zajęców x i wilków y

- Populacja y zmienia się w czasie przez reprodukcje zależnie od populacji x

Populacje zajęców x i wilków y

- Populacja y zmienia się w czasie przez reprodukcje zależnie od populacji x $\frac{dy}{dt} = dxy$

Równania różniczkowe Lotka-Volterra

Różnice drapieżników: $\frac{dy}{dt} = dxy\ -$

Różnice zajęców: $\frac{dx}{dt} =$

Równania różniczkowe Lotka-Volterra

Różnice drapieżników: $\frac{dy}{dt} = dxy\ - cy$

Różnice zajęców: $\frac{dx}{dt} =$

Równania różniczkowe Lotka-Volterra

Różnice drapieżników: $\frac{dy}{dt} = dxy\ - cy$

Różnice zajęców: $\frac{dx}{dt} = ax\ -$

Równania różniczkowe Lotka-Volterra

Różnice drapieżników: $\frac{dy}{dt} = dxy\ - cy$

Różnice zajęców: $\frac{dx}{dt} = ax\ - bxy$

- Gdy $y = 0, x' = ax$ → wzrost wykładniczy

- Gdy $y = 0, x' = ax$ → wzrost wykładniczy
- Punkty stacjonarne:
- $(x, y) = (0,0)$

- Gdy $y = 0, x' = ax$ → wzrost wykładniczy

- Punkty stacjonarne:

$$0 = dxy - cy$$
$$0 = ax - bxy$$

- $(x, y) = (0,0)$

- Gdy $y = 0, x' = ax$ → wzrost wykładniczy

- Punkty stacjonarne:

$$0 = dxy - cy$$
$$0 = ax - bxy$$

- $(x, y) = (0,0)$

$$a - by = 0$$
$$y = \frac{a}{b}$$

- Gdy $y = 0, x' = ax$ → wzrost wykładniczy

- Punkty stacjonarne:

$$0 = dxy - cy$$
$$0 = ax - bxy$$

- $(x, y) = (0,0)$

$$a - by = 0$$
$$y = \frac{a}{b}$$

$$-c + dx = 0$$
$$x = \frac{c}{d}$$

- Gdy $y = 0, x' = ax$ → wzrost wykładniczy

- Punkty stacjonarne:

$$0 = dxy - cy$$
$$0 = ax - bxy$$

- $(x, y) = (0,0)$

$$a - by = 0$$
$$y = \frac{a}{b}$$

$$-c + dx = 0$$
$$x = \frac{c}{d}$$

$$(x, y) = (\frac{c}{d}, \frac{a}{b})$$

Jacobian

$$\frac{dy}{dt} = dxy - cy$$
$$\frac{dx}{dt} = ax - bxy$$

- $J =$

Jacobian

$$\frac{dy}{dt} = dxy - cy$$
$$\frac{dx}{dt} = ax - bxy$$

- $J = \begin{pmatrix} a - by & \\ & \end{pmatrix}$

Jacobian

$$\frac{dy}{dt} = dxy - cy$$
$$\frac{dx}{dt} = ax - bxy$$

- $J = \begin{pmatrix} a - by & -bx \end{pmatrix}$

Jacobian

$$\frac{dy}{dt} = dxy - cy$$
$$\frac{dx}{dt} = ax - bxy$$

- $J = \begin{pmatrix} a - by & -bx \\ dy & \end{pmatrix}$

Jacobian

$$\frac{dy}{dt} = dxy - cy$$
$$\frac{dx}{dt} = ax - bxy$$

- $J = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$

Jacobian

$(x, y) = (0,0)$

- $J = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$

Jacobian

$(x, y) = (0,0)$

- $J = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$

- $\lambda_1 = a \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $\lambda_2 = -c \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

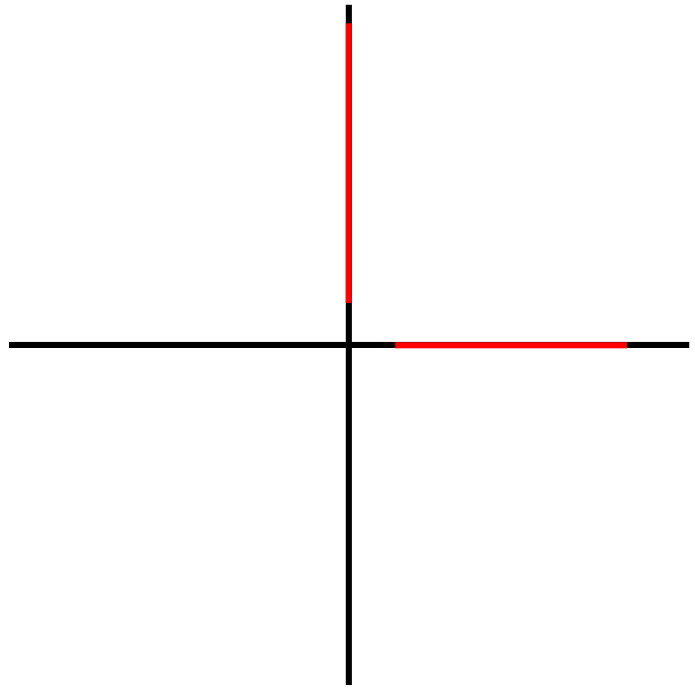
Jacobian

$(x, y) = (0,0)$

- $J = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$

- $\lambda_1 = a \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $\lambda_2 = -c \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



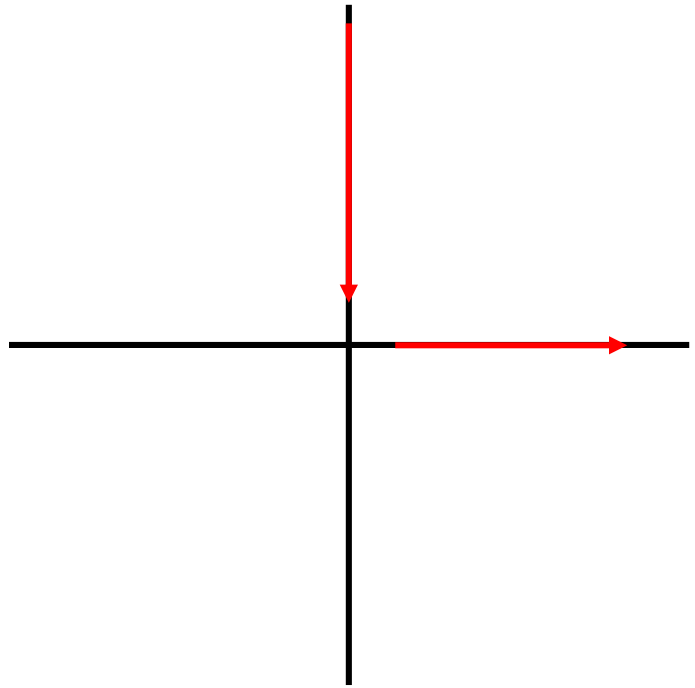
Jacobian

$(x, y) = (0,0)$

- $J = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$

- $\lambda_1 = a \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $\lambda_2 = -c \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



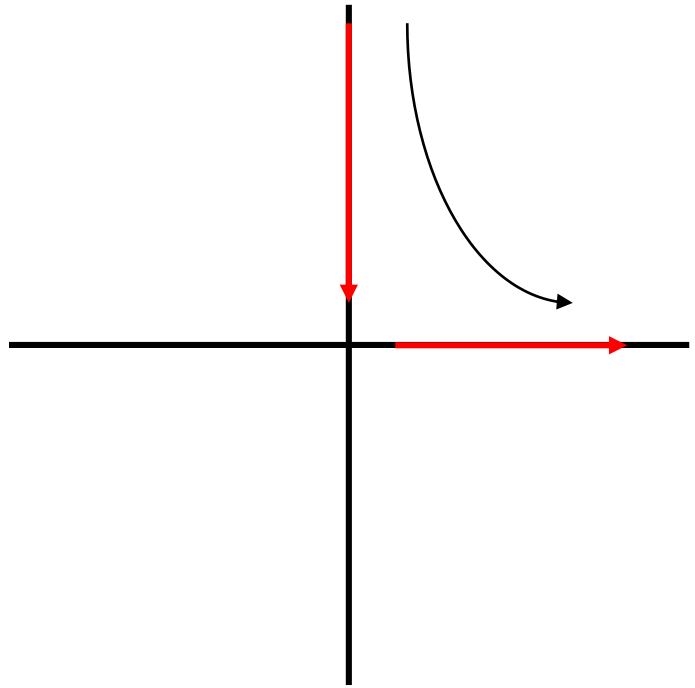
Jacobian

$(x, y) = (0,0)$

- $J = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$

- $\lambda_1 = a \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $\lambda_2 = -c \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



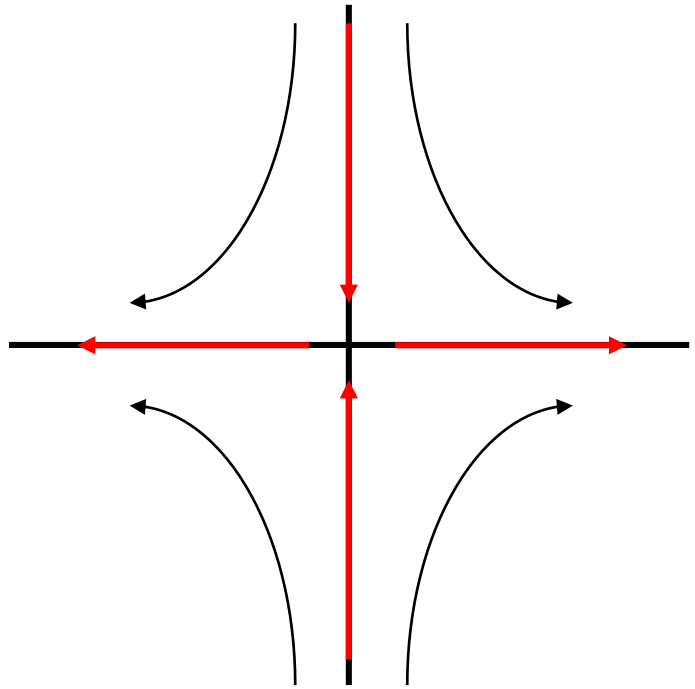
Jacobian

$(x, y) = (0,0)$

- $J = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$

- $\lambda_1 = a \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $\lambda_2 = -c \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Jacobian

$$(x, y) = (\frac{c}{d}, \frac{a}{b})$$

- $J = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}$

Jacobian

$$(x, y) = (\frac{c}{d}, \frac{a}{b})$$

- $J = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}$
- $\begin{vmatrix} -\lambda & -\frac{bc}{d} \\ \frac{da}{b} & -\lambda \end{vmatrix}$

Jacobian

$$(x, y) = (\frac{c}{d}, \frac{a}{b})$$

- $J = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}$
- $\begin{vmatrix} -\lambda & -\frac{bc}{d} \\ \frac{da}{b} & -\lambda \end{vmatrix} = \lambda^2 + ac = 0$

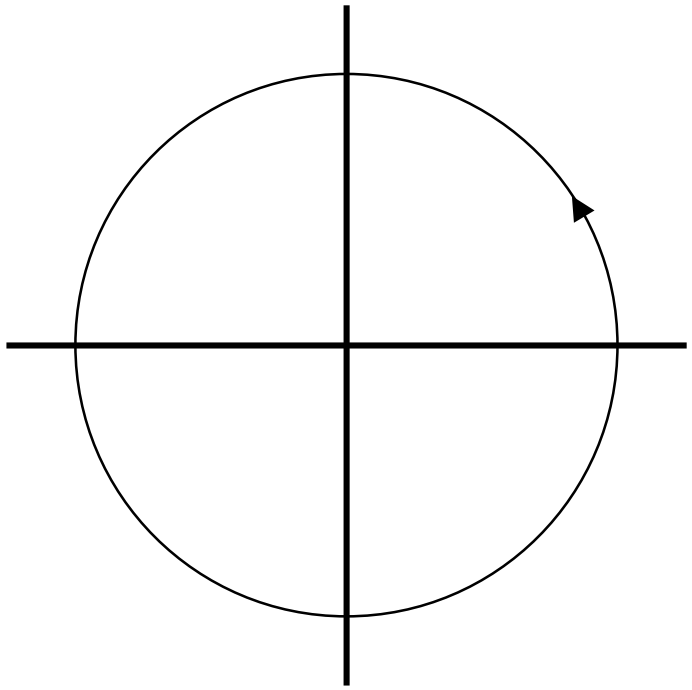
Jacobian

$$(x, y) = (\frac{c}{d}, \frac{a}{b})$$

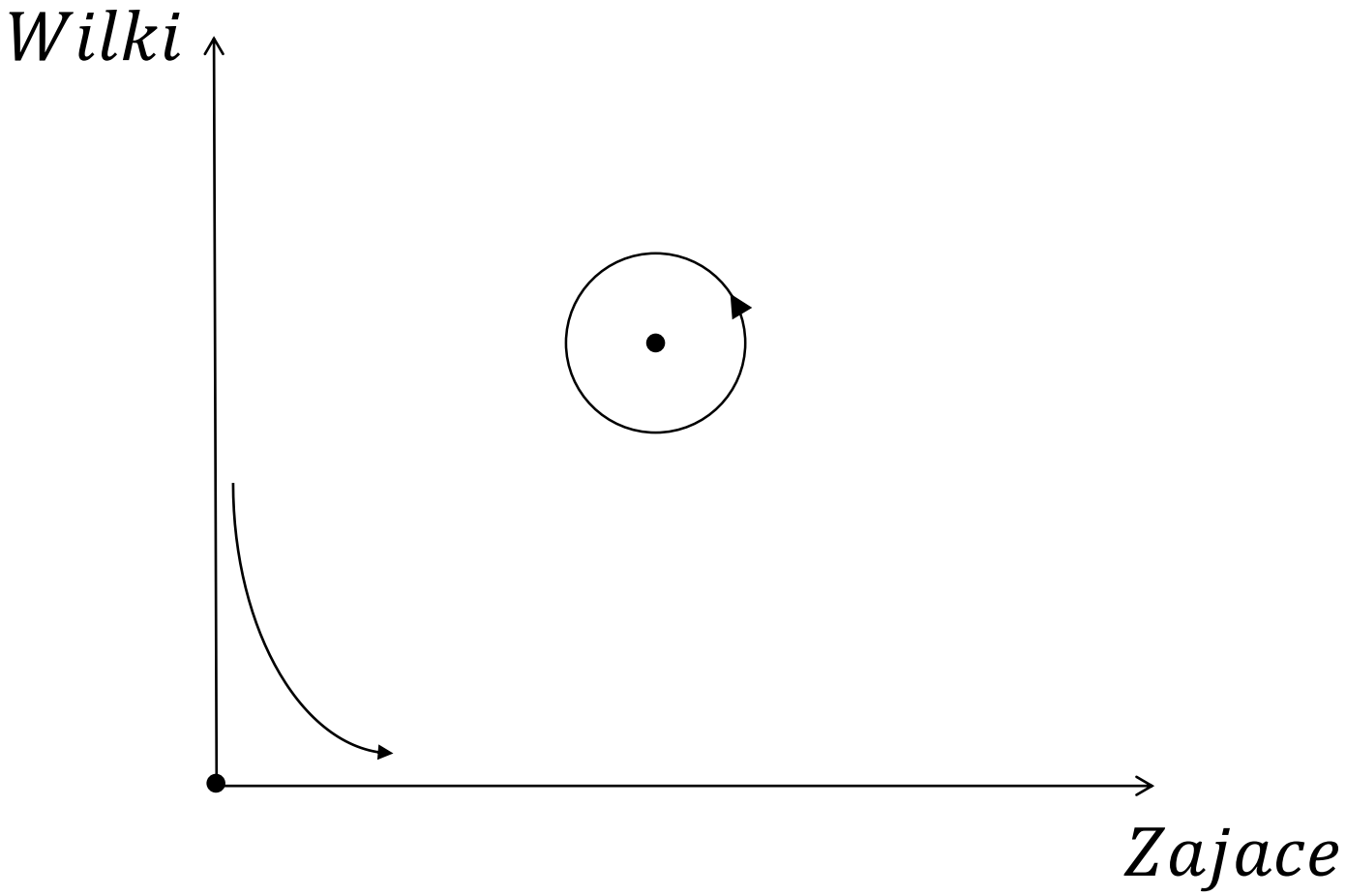
- $J = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}$
- $\begin{vmatrix} -\lambda & -\frac{bc}{d} \\ \frac{da}{b} & -\lambda \end{vmatrix} = \lambda^2 + ac = 0$
- $\lambda = \pm\sqrt{ac}i$

Jacobian

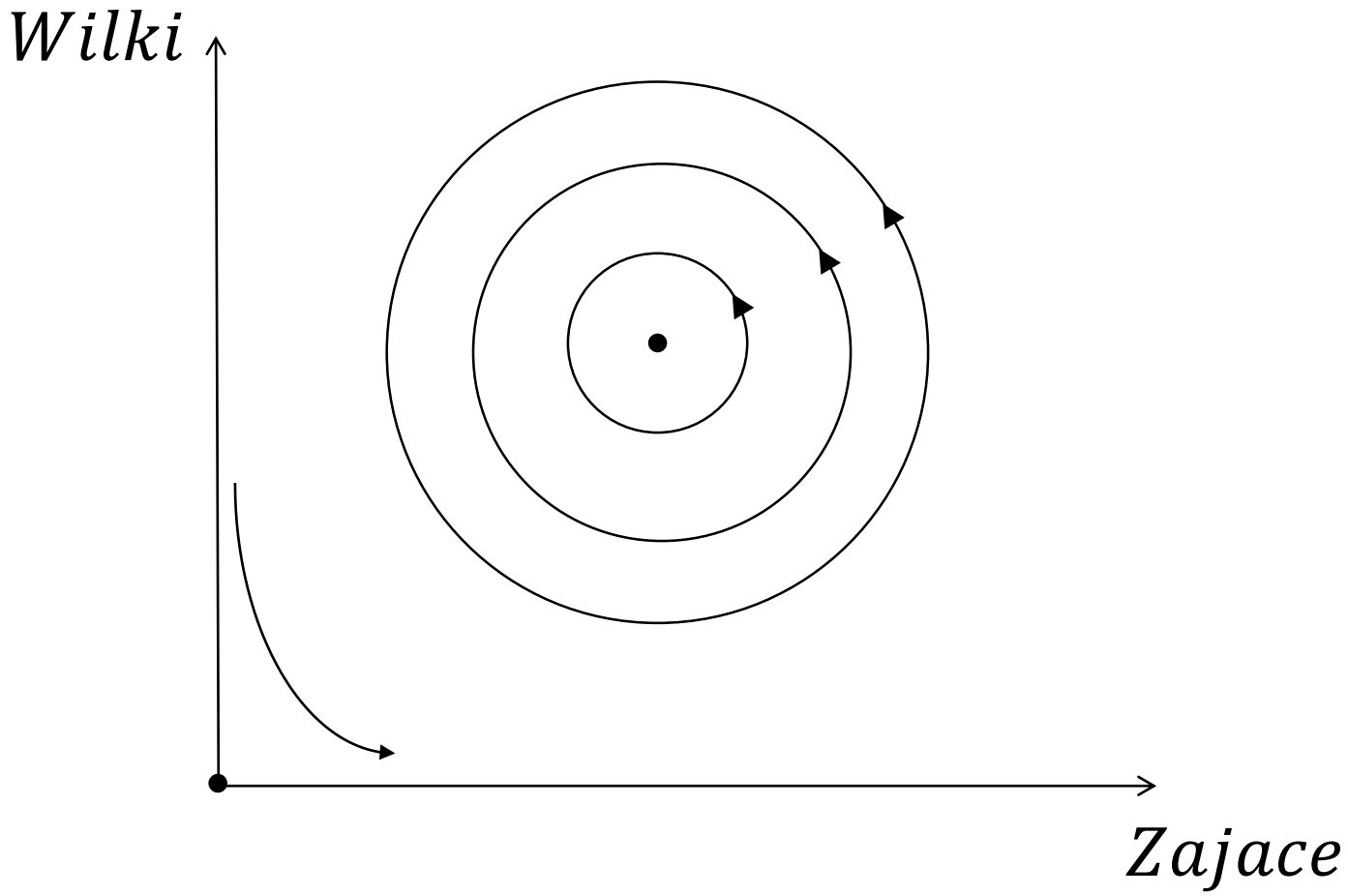
- $J = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}$
- $\begin{vmatrix} -\lambda & -\frac{bc}{d} \\ \frac{da}{b} & -\lambda \end{vmatrix} = \lambda^2 + ac = 0$
- $\lambda = \pm\sqrt{ac}i$



Analiza



Analiza



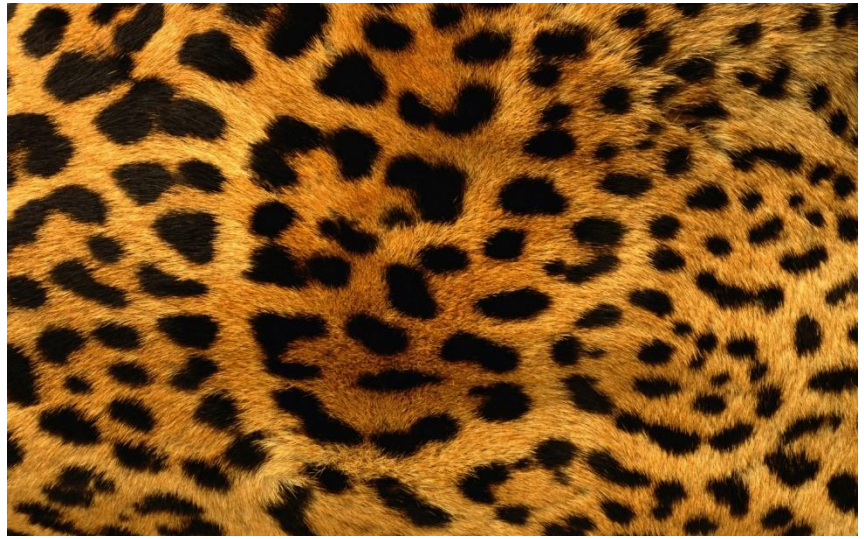
Zadanie

- Zaimplementuj model Lotki-Volterry i drapieżnik-ofiara. Wyświetl obraz fazowy i wykresy populacji jako funkcje czasu.

Zadanie

- Rozwiń model drapieżnik-ofiara o wzrost logistyczny dla zajęcy. Przeprowadź wstępna analizę tego systemu. Zasymuluj nowy model i wyświetl obraz fazowy i pole wektorowe. Co da się zauważyć?

Modelowanie wzorców biologicznych za pomocy równań różniczkowych

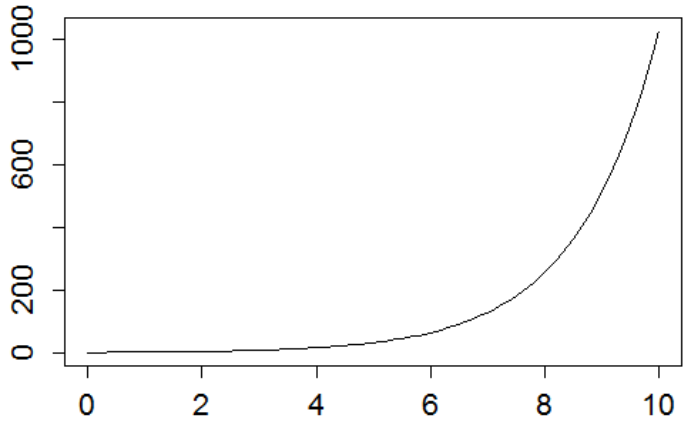


Równania reakcja-dyfuzja

$$\frac{\partial c}{\partial t} = \underbrace{\mathrm{D}\frac{\partial^2 c}{\partial x^2}}_{\text{Dyfuzja}} + \underbrace{f(c)}_{\text{Reakcja}}$$

Reakcja w 1D?

- Na przykład model wzrostu populacji
- $\frac{dc}{dt} = Kc$
- $\mathrm{c}(t) = C_0 e^{Kt}, C_0 = c(0)$

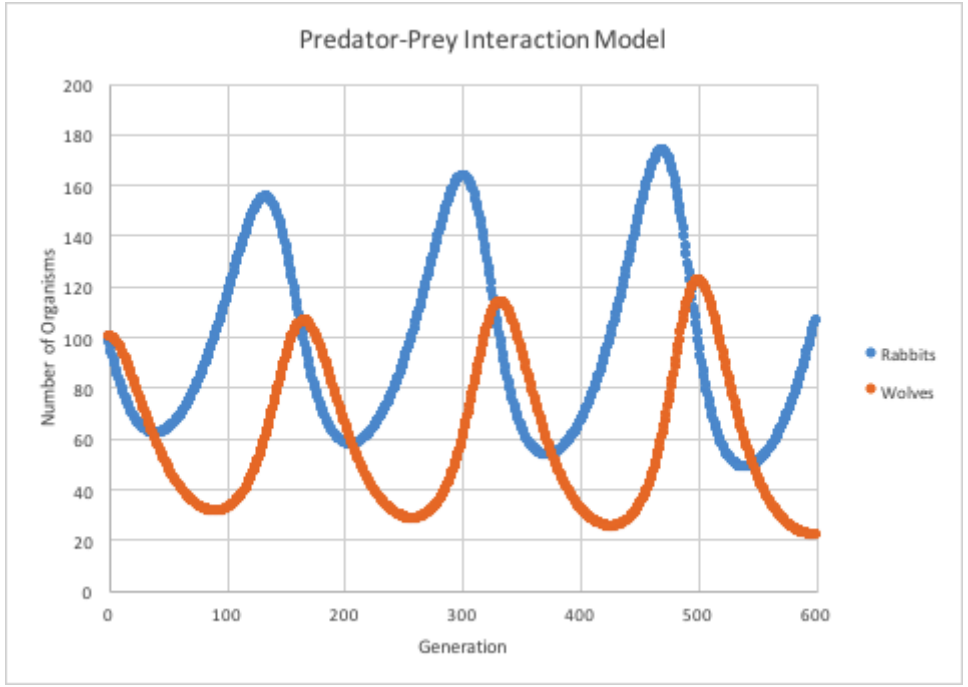


Reakcja 2D?

Lotka-Volterra:

$$\frac{dy}{dt} = dxy - cy$$

$$\frac{dx}{dt} = ax - bxy$$

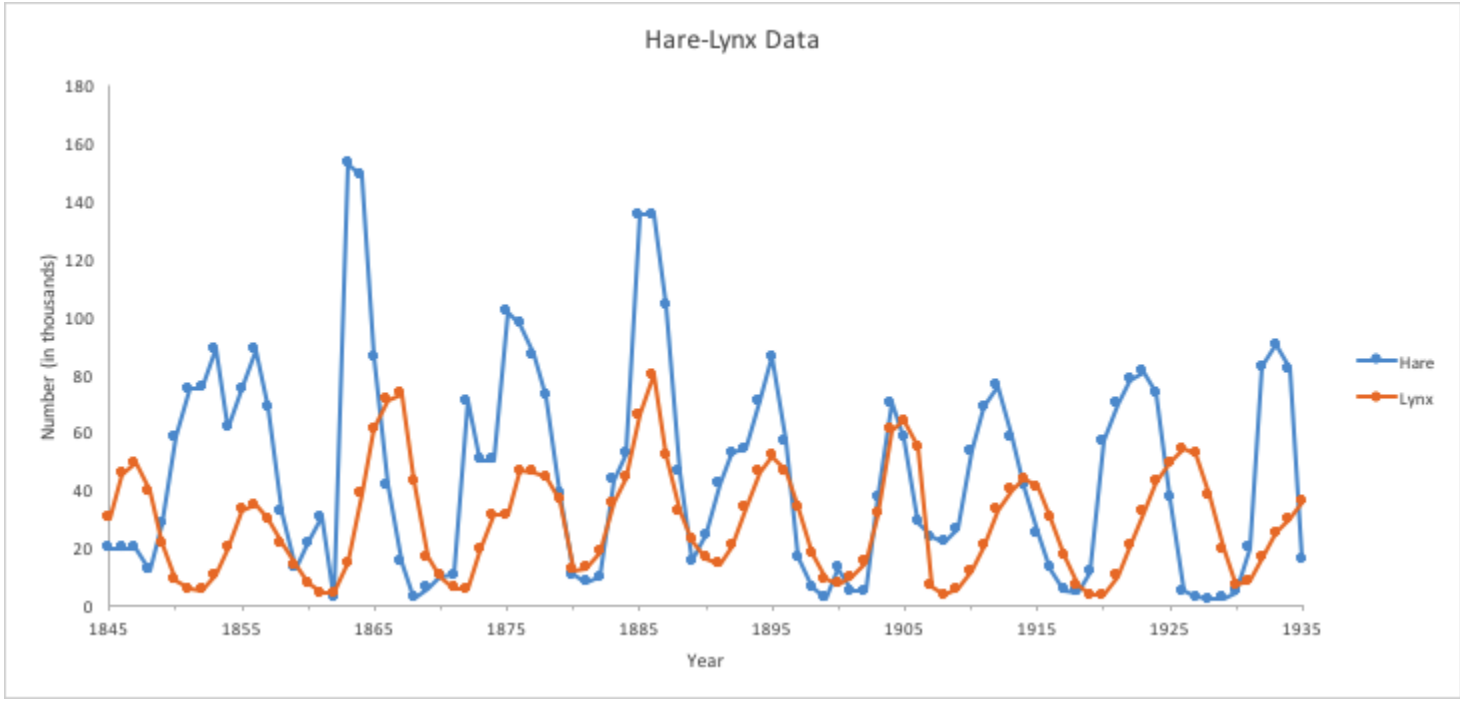


Reakcja 2D?

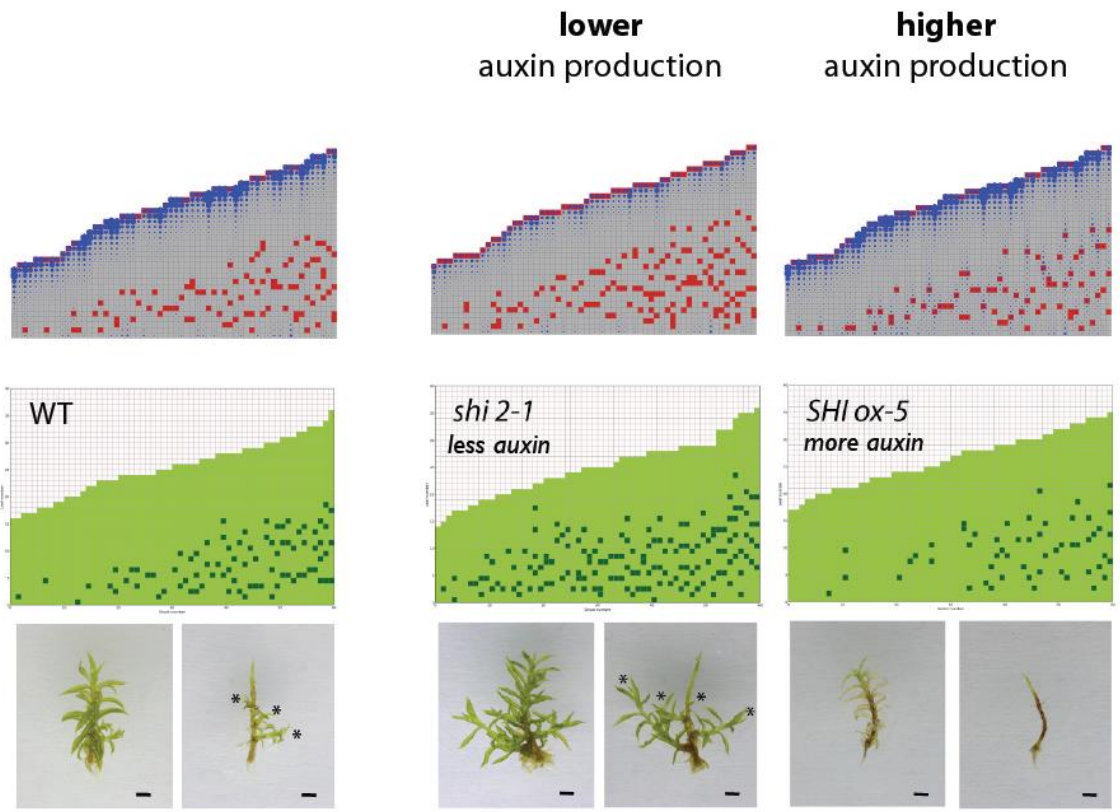
Lotka-Volterra:

$$\frac{dy}{dt} = dxy - cy$$

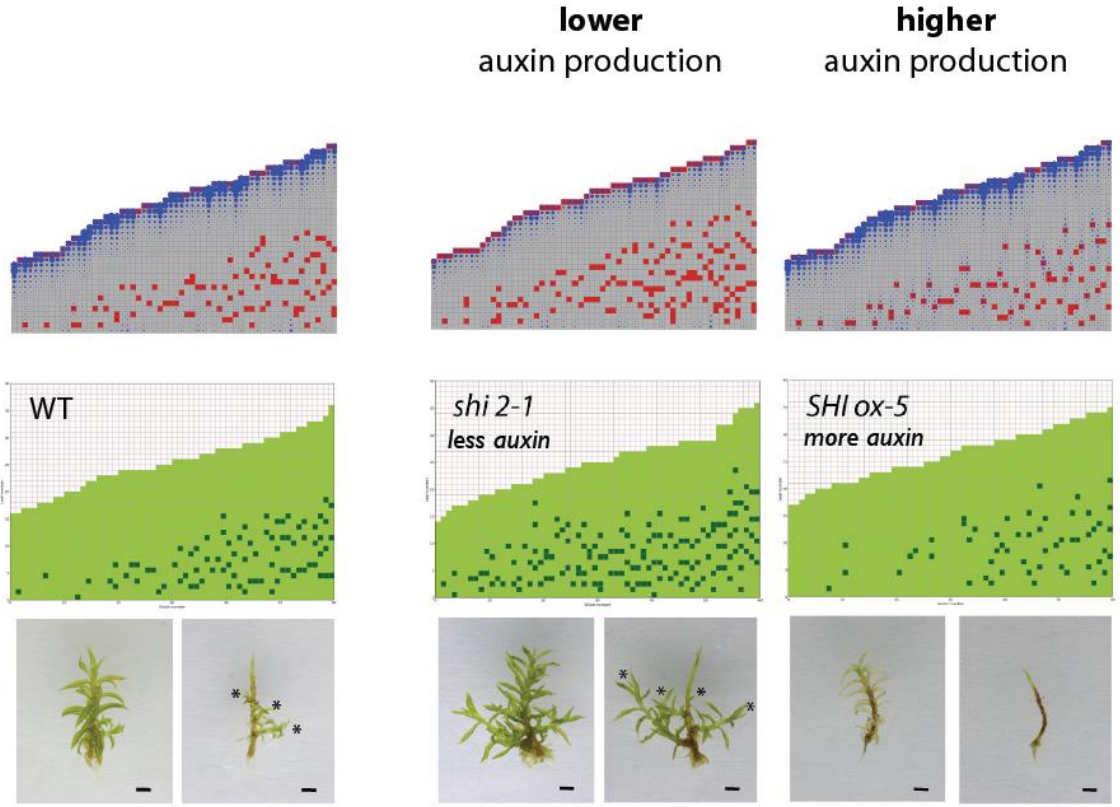
$$\frac{dx}{dt} = ax - bxy$$



Zróżnicowanie tkanki biologicznej



Zróżnicowanie tkanki biologicznej



$$\frac{\partial c}{\partial t} = \mathrm{D}\frac{\partial^2 c}{\partial x^2} - \frac{1}{\tau}c + \rho$$

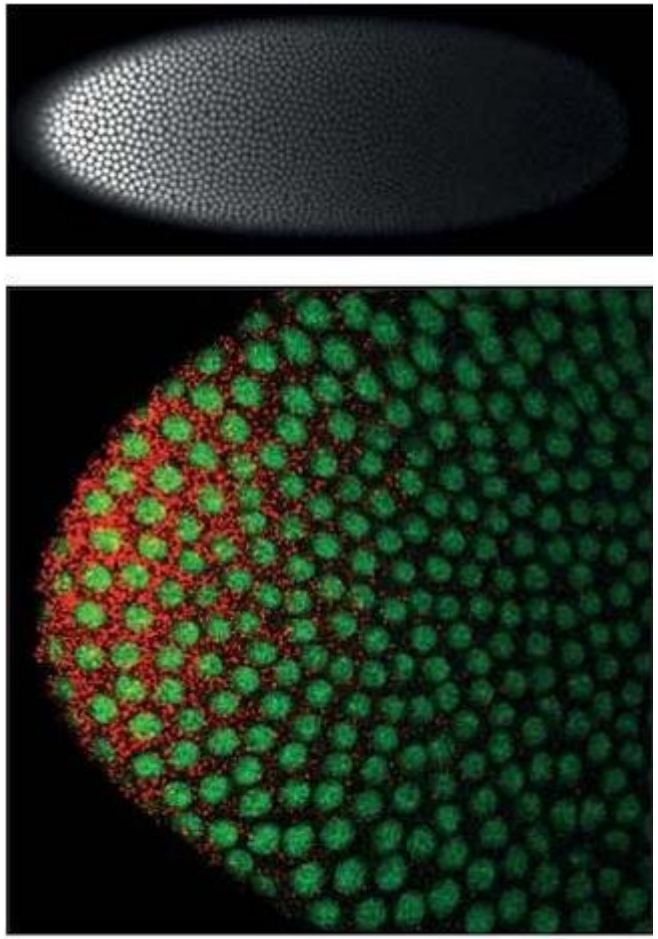
Degradacja

Produkcja

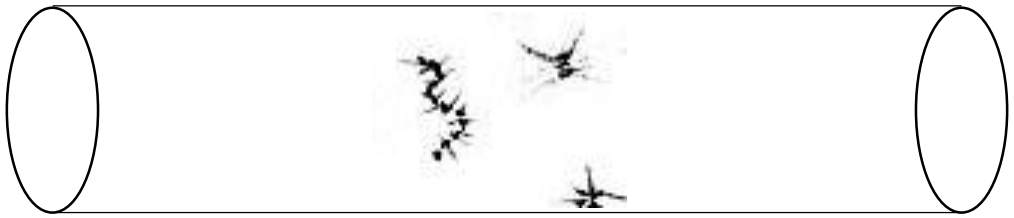
Dyfuzja

Bicoid

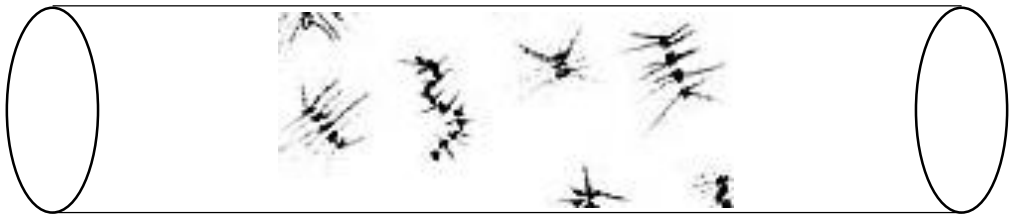
- Wykladniczy gradient białka w jajku Drosophila (Driever et al. 1988, Cell)



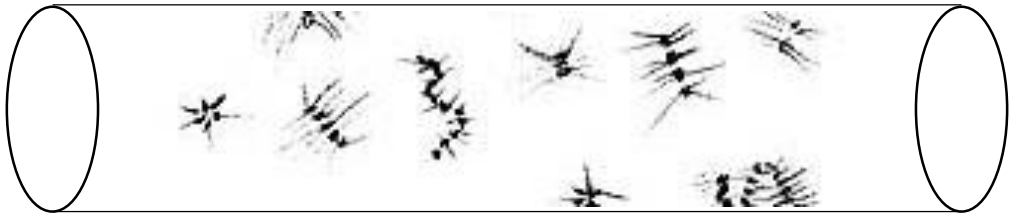
Plankton



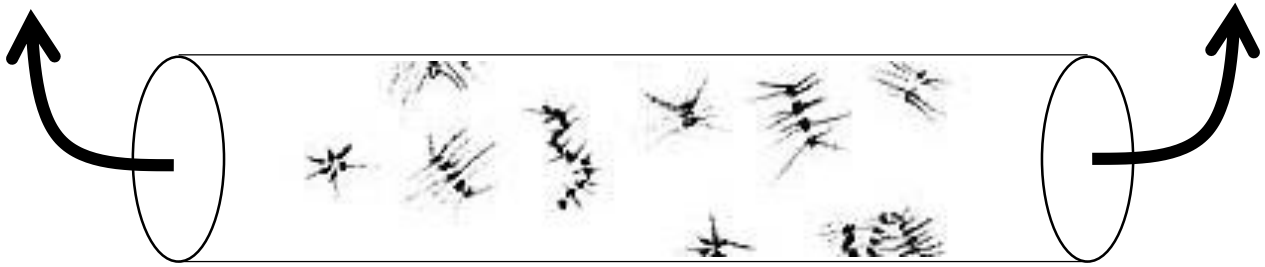
Plankton



Plankton



Plankton



Plankton

$$\frac{\partial c}{\partial t} = \mathrm{D}\frac{\partial^2 c}{\partial x^2} + Kc$$

Plankton

$$\frac{\partial c}{\partial t} = \mathrm{D}\frac{\partial^2 c}{\partial x^2} + Kc$$

$$c(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(K - \frac{n^2\pi^2 D}{L^2}\right)t}$$

Plankton

$$\frac{\partial c}{\partial t} = \mathrm{D}\frac{\partial^2 c}{\partial x^2} + Kc$$

$$c(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(K - \frac{n^2\pi^2 D}{L^2}\right)t}$$

$$B_n = \frac{2}{L}\int_0^L C_0 \sin\left(\frac{n\pi x}{L}\right) dx$$

↑

Współczynnik Fouriera

n = stopień

Dla n=1

$$c(x,t) = B_1 \sin(\frac{\pi x}{L}) e^{\left(K - \frac{\pi^2 D}{L^2}\right)t}$$

Dla n=1

$$c(x,t) = B_1 \sin\left(\frac{\pi x}{L}\right) e^{\left(K - \frac{\pi^2 D}{L^2}\right)t}$$

$$K - \frac{\pi^2 D}{L^2} < 0$$

Populacja się zmniejsza

Dla n=1

$$c(x,t) = B_1 \sin\left(\frac{\pi x}{L}\right) e^{\left(K - \frac{\pi^2 D}{L^2}\right)t}$$

$K - \frac{\pi^2 D}{L^2} < 0$ Populacja się zmniejsza

$K - \frac{\pi^2 D}{L^2} > 0$ Populacja się zwiększa

- $K - \frac{\pi^2 D}{L^2} = 0$

- $K - \frac{\pi^2 D}{L^2} = 0$
- $L = \pi\sqrt{\frac{D}{K}}$

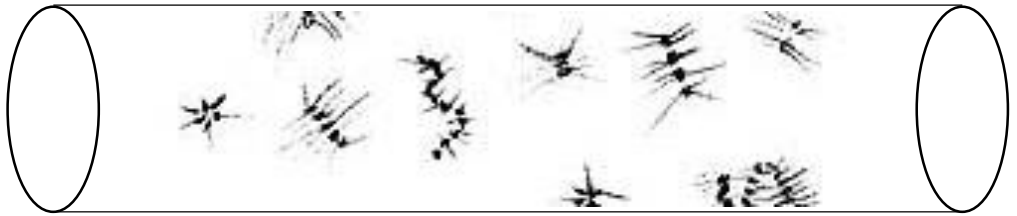
- $K - \frac{\pi^2 D}{L^2} = 0$
- $L = \pi\sqrt{\frac{D}{K}}$
- $L_c = \pi\sqrt{\frac{D}{K}}$

L_c = krytyczna długość

- $K - \frac{\pi^2 D}{L^2} = 0$
- $L = \pi\sqrt{\frac{D}{K}}$
- $L_c = \pi\sqrt{\frac{D}{K}}$

L_c = krytyczna długość

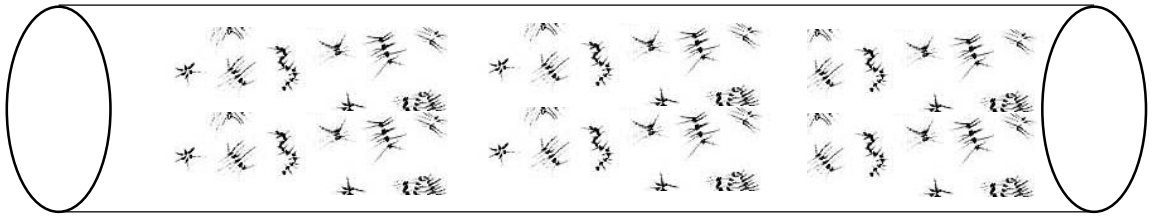
$L = L_c$



- $K - \frac{\pi^2 D}{L^2} = 0$
- $L = \pi\sqrt{\frac{D}{K}}$
- $L_c = \pi\sqrt{\frac{D}{K}}$

L_c = krytyczna długość

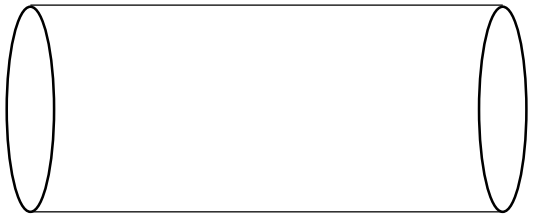
$L > L_c$



- $K - \frac{\pi^2 D}{L^2} = 0$
- $L = \pi\sqrt{\frac{D}{K}}$
- $L_c = \pi\sqrt{\frac{D}{K}}$

L_c = krytyczna długość

$L < L_c$



Model reakcji-dyfuzji i przestrzenna dziedzina

- $\frac{\partial c}{\partial t} = \mathrm{D}\frac{\partial^2 c}{\partial x^2} + Kc$

- $L_c = \pi\sqrt{\frac{D}{K}}$

Wzorce Turinga

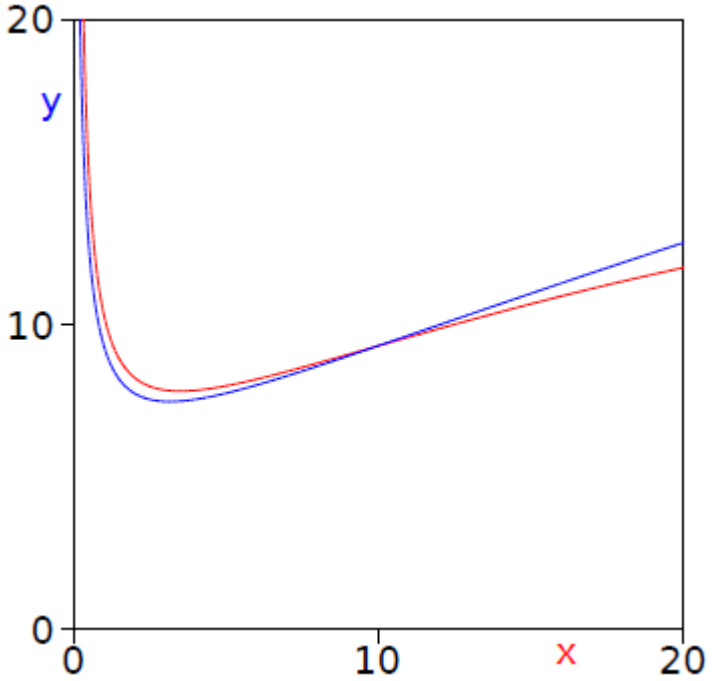
- W roku 1952 Alan Turing teoretycznie pokazał ze punkt stacjonarny stabilny staje się niestabilny jako wynik dyfuzji
- Wynik nie jest intuicyjny bo dyfuzja osobno zmniejsza poziomy niejednorodności

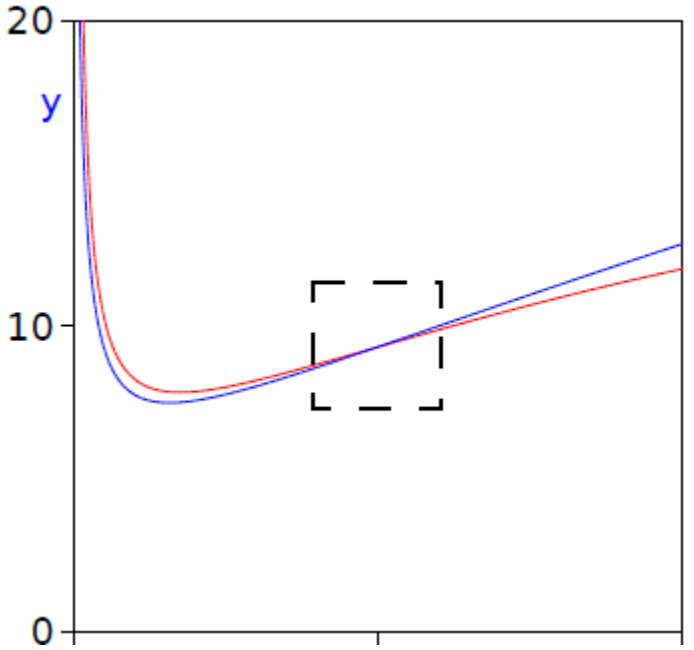
Niestabilność Turinga

$$\begin{aligned} \frac{\partial x}{\partial t} &= f(x,y) + D_a \Delta x \\ \frac{\partial y}{\partial t} &= g(x,y) + D_i \Delta y \end{aligned}$$

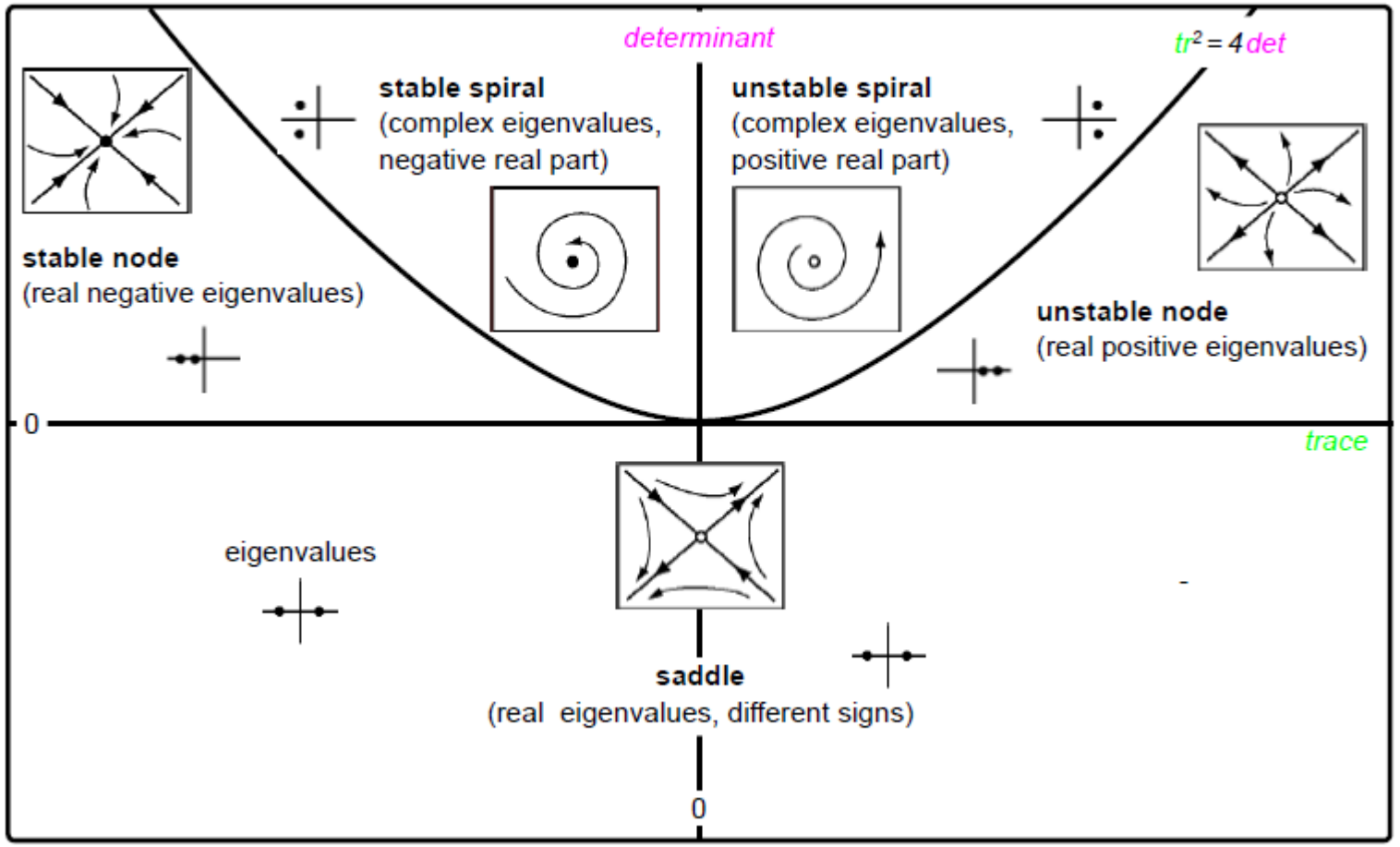
$$\begin{aligned} f(x^*,y^*) &= 0 \\ g(x^*,y^*) &= 0 \end{aligned}$$

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$





Stabilność punktu stacjonarnego



Warunki stabilności bez dyfuzji:

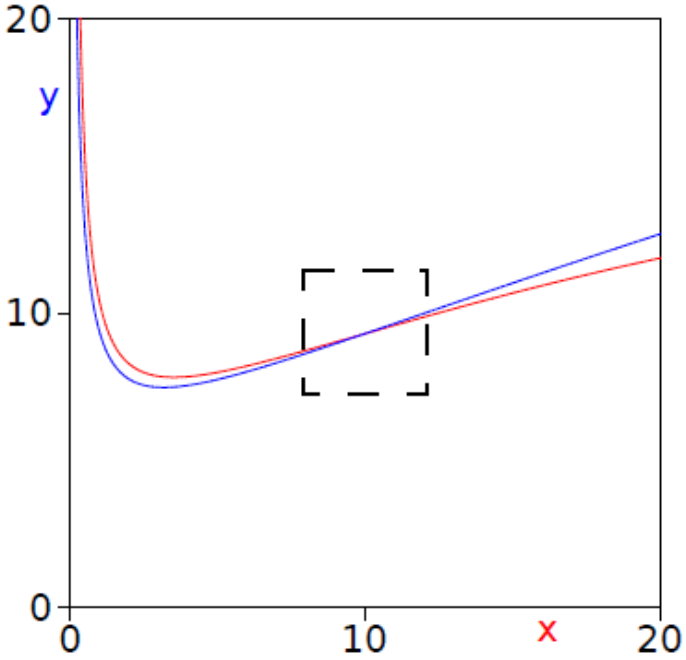
$$a_{11} + a_{22} < 0$$
$$a_{11} \cdot a_{22} - a_{21} \cdot a_{12} > 0$$

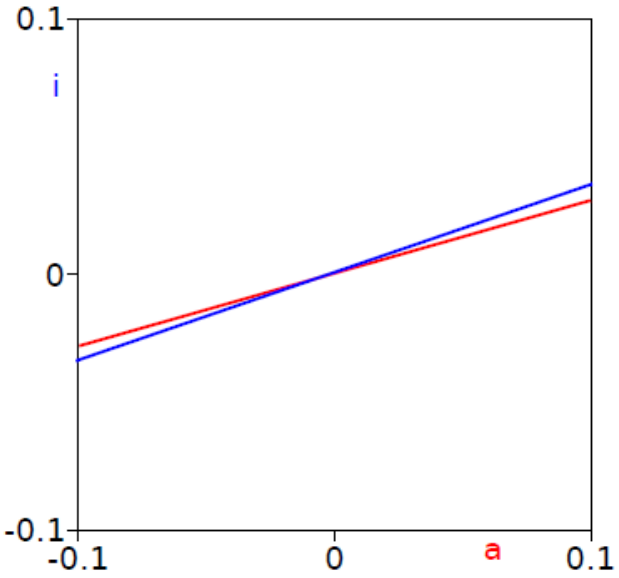
Niestabilność Turinga

$$\begin{aligned} a &= x - x^* \\ i &= y - y^* \end{aligned} \quad ; \quad \begin{aligned} x &= x^* + a \\ y &= y^* + i \end{aligned}$$

$$\begin{aligned} \frac{\partial a}{\partial t} &= D_a \Delta a + f(x^* + a, y^* + i) \\ \frac{\partial i}{\partial t} &= D_i \Delta i + g(x^* + a, y^* + i) \end{aligned}$$

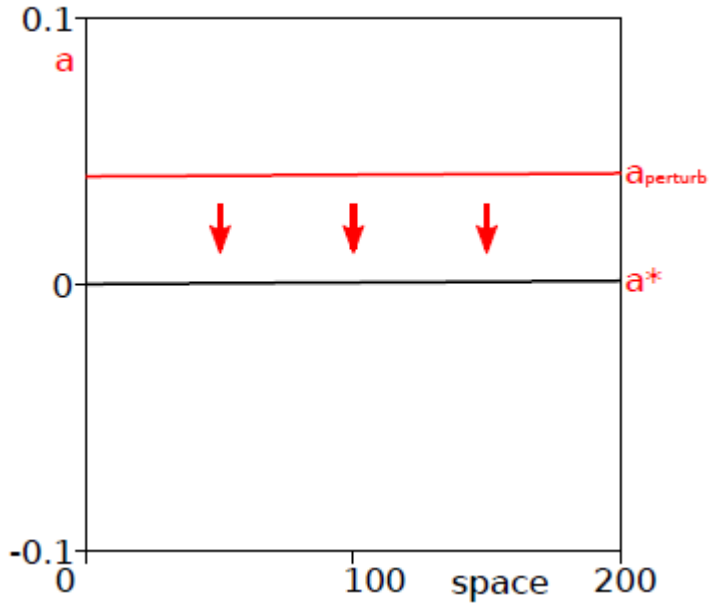
$$\begin{aligned} \frac{\partial a}{\partial t} &= D_a \Delta a + a_{11} a + a_{12} i \\ \frac{\partial i}{\partial t} &= D_i \Delta i + a_{21} a + a_{22} i \end{aligned}$$





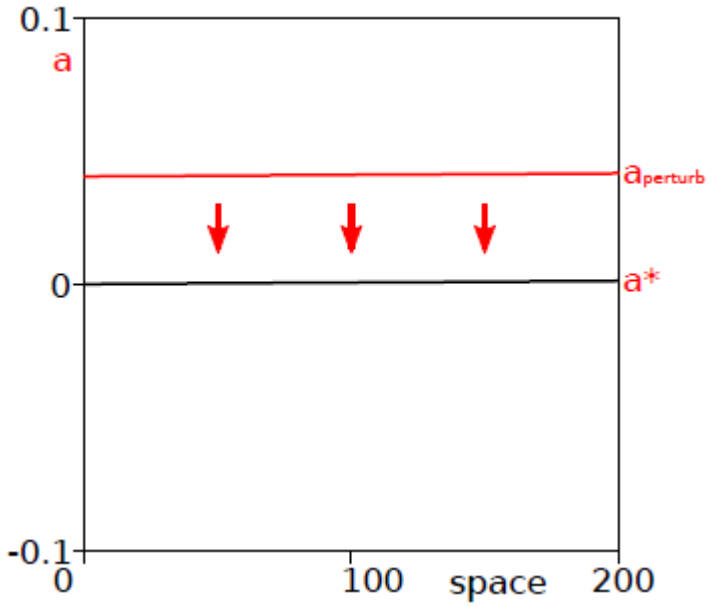
Metoda

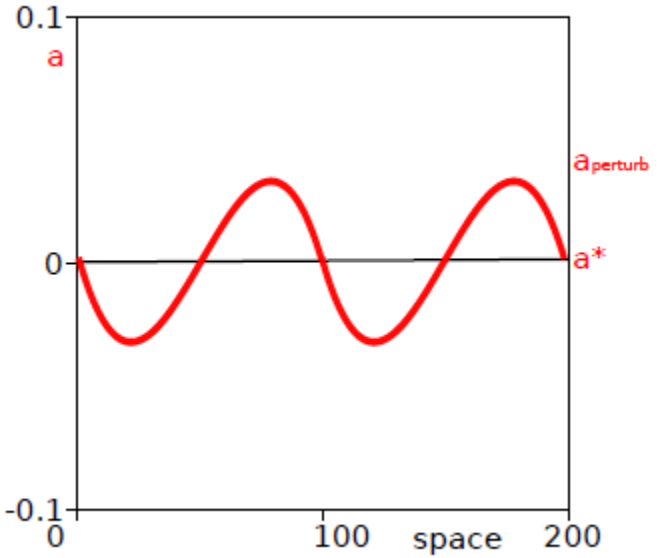
- Po jednorodnej perturbacji system wraca do równowagi



Metoda

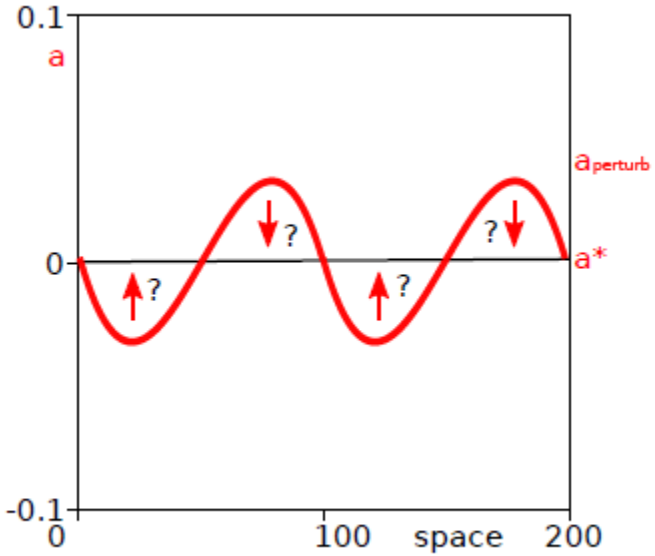
- Po jednorodnej perturbacji system wraca do równowagi
- Co się stanie po niejednorodnej perturbacji?





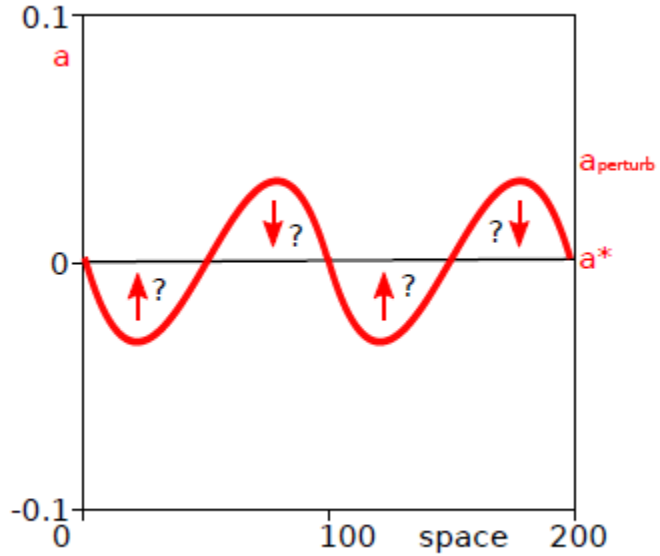
Metoda

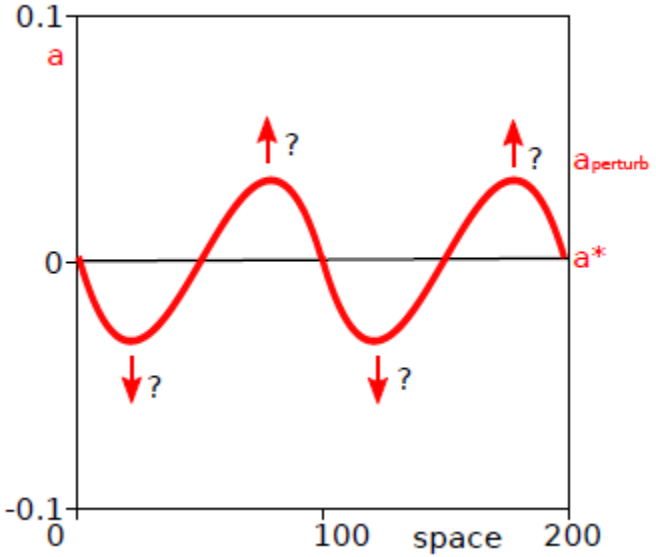
- Założenie: gdy perturbacja jest sinusoidalna to w trakcie perturbacji pozostanie taka



Metoda

- Założenie: gdy perturbacja jest sinusoidalna to w trakcie perturbacji pozostanie taka
- Gdy to prawda, to pytanie o stabilność sprowadza się do tego czy amplituda się zwiększa czy dąży do zera





Aproksymacja cos

$$
\begin{aligned}
a(x,t) &= A(t)\cos(kx) \\
i(x,t) &= I(t)\cos(kx)
\end{aligned}
$$

Aproksymacja cos

$$
\begin{aligned}
a(x,t) &= A(t)\cos(kx) \\
i(x,t) &= I(t)\cos(kx)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(A\cos(kx))}{\partial t} &= D_a\Delta(A\cos(kx)) + a_{11}(A\cos(kx)) + a_{12}(I\cos(kx)) \\
\frac{\partial(I\cos(kx))}{\partial t} &= D_i\Delta(I\cos(kx)) + a_{21}(A\cos(kx)) + a_{22}(I\cos(kx))
\end{aligned}
$$

Aproksymacja cos

$$
\begin{aligned}
a(x,t) &= A(t)\cos(kx) \\
i(x,t) &= I(t)\cos(kx)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(A\cos(kx))}{\partial t} &= D_a\Delta(A\cos(kx)) + a_{11}(A\cos(kx)) + a_{12}(I\cos(kx)) \\
\frac{\partial(I\cos(kx))}{\partial t} &= D_i\Delta(I\cos(kx)) + a_{21}(A\cos(kx)) + a_{22}(I\cos(kx))
\end{aligned}
$$

$$
\begin{aligned}
\cos(kx)\frac{\partial A}{\partial t} &= -k^2 D_a A\cos(kx) + a_{11}\cos(kx)A + a_{12}\cos(kx)I \\
\cos(kx)\frac{\partial I}{\partial t} &= -k^2 D_i I\cos(kx) + a_{21}\cos(kx)A + a_{22}\cos(kx)I
\end{aligned}
$$

Aproksymacja cos

$$
\begin{aligned}
a(x,t) &= A(t)\cos(kx) \\
i(x,t) &= I(t)\cos(kx)
\end{aligned}
$$

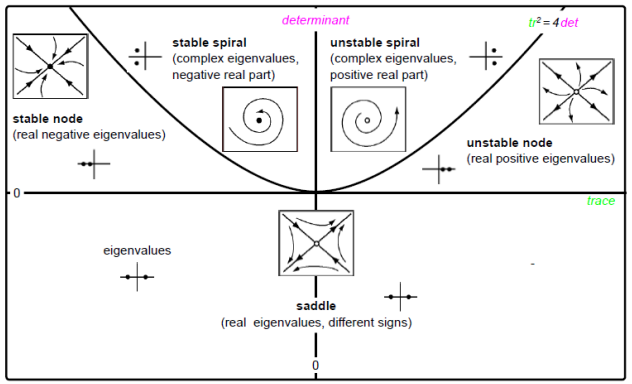
$$
\begin{aligned}
\frac{\partial(A\cos(kx))}{\partial t} &= D_a\Delta(A\cos(kx)) + a_{11}(A\cos(kx)) + a_{12}(I\cos(kx)) \\
\frac{\partial(I\cos(kx))}{\partial t} &= D_i\Delta(I\cos(kx)) + a_{21}(A\cos(kx)) + a_{22}(I\cos(kx))
\end{aligned}
$$

$$
\begin{aligned}
\cos(kx)\frac{\partial A}{\partial t} &= -k^2 D_a A\cos(kx) + a_{11}\cos(kx)A + a_{12}\cos(kx)I \\
\cos(kx)\frac{\partial I}{\partial t} &= -k^2 D_i I\cos(kx) + a_{21}\cos(kx)A + a_{22}\cos(kx)I
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A}{\partial t} &= -k^2 D_a A + a_{11}A + a_{12}I \\
\frac{\partial I}{\partial t} &= -k^2 D_i I + a_{21}A + a_{22}I
\end{aligned}
$$

Macierz Jacobiego

$$J = \begin{pmatrix} -D_a k^2 + a_{11} & a_{12} \\ a_{21} & -D_i k^2 + a_{22} \end{pmatrix}$$



$$trJ = -D_a k^2 - D_i k^2 + a_{11} + a_{22} < 0$$

$$(-D_a k^2 + a_{11}) \cdot (-D_i k^2 + a_{22}) - a_{12} \cdot a_{21} < 0$$

$$\begin{aligned} (-D_a \kappa + a_{11}) \cdot (-D_i \kappa + a_{22}) - a_{12} \cdot a_{21} &< 0 \\ D_a D_i \kappa^2 - (D_a a_{22} + D_i a_{11}) \kappa + a_{11} \cdot a_{22} - a_{21} \cdot a_{12} &< 0 \end{aligned}$$

Macierz Jacobiego

$$\frac{\partial detJ}{\partial \kappa} = 2D_a D_i \kappa - D_a a_{22} - D_i a_{11}$$

Macierz Jacobiego

$$\frac{\partial detJ}{\partial \kappa} = 2D_a D_i \kappa - D_a a_{22} - D_i a_{11}$$

$$2D_a D_i \kappa_{min} - D_a a_{22} - D_i a_{11} = 0$$

$$\kappa_{min} = \frac{D_a a_{22} + D_i a_{11}}{2D_a D_i}$$

Macierz Jacobiego

$$\frac{\partial detJ}{\partial \kappa} = 2D_aD_i\kappa - D_aa_{22} - D_ia_{11}$$

$$2D_aD_i\kappa_{min} - D_aa_{22} - D_ia_{11} = 0$$

$$\kappa_{min} = \frac{D_aa_{22}+D_ia_{11}}{2D_aD_i}$$

$$\begin{aligned}
detJ_{min} &= D_aD_i\kappa_{min}^2 - (D_aa_{22} + D_ia_{11})\kappa_{min} + a_{11}\cdot a_{22} - a_{21}\cdot a_{12} \\
detJ_{min} &= \frac{(D_aa_{22}+D_ia_{11})^2}{4D_aD_i} - \frac{(D_aa_{22}+D_ia_{11})^2}{2D_aD_i} + a_{11}\cdot a_{22} - a_{21}\cdot a_{12} \\
detJ_{min} &= -\frac{(D_aa_{22}+D_ia_{11})^2}{4D_aD_i} + a_{11}\cdot a_{22} - a_{21}\cdot a_{12}
\end{aligned}$$

Minimum detJ

$$
\begin{aligned}
detJ_{min} &< 0 \\
\frac{(D_a a_{22} + D_i a_{11})^2}{4 D_a D_i} &> a_{11} \cdot a_{22} - a_{21} \cdot a_{12} \\
D_a a_{22} + D_i a_{11} &> 2\sqrt{D_a D_i (a_{11} \cdot a_{22} - a_{21} \cdot a_{12})}
\end{aligned}
$$

$$
\begin{aligned}
&a_{11} + a_{22} < 0 \\
&a_{11} \cdot a_{22} - a_{21} \cdot a_{12} > 0 \\
&D_a a_{22} + D_i a_{11} > 2\sqrt{D_a D_i (a_{11} \cdot a_{22} - a_{21} \cdot a_{12})} > 0
\end{aligned}
$$

$$
D_a a_{22} + D_i a_{11} > 0
$$

Wynik Analizy

$$-D_a\,|a_{22}| + D_i\,|a_{11}| > 0$$

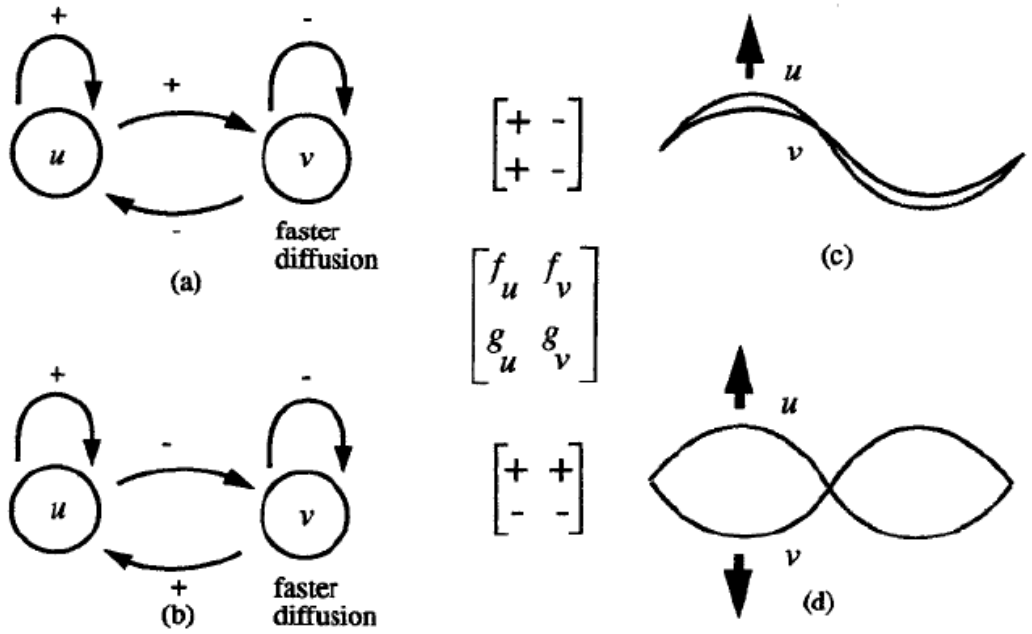
$$\frac{D_i}{|a_{22}|} > \frac{D_a}{|a_{11}|}$$

Możliwe Macierze Jacobiego

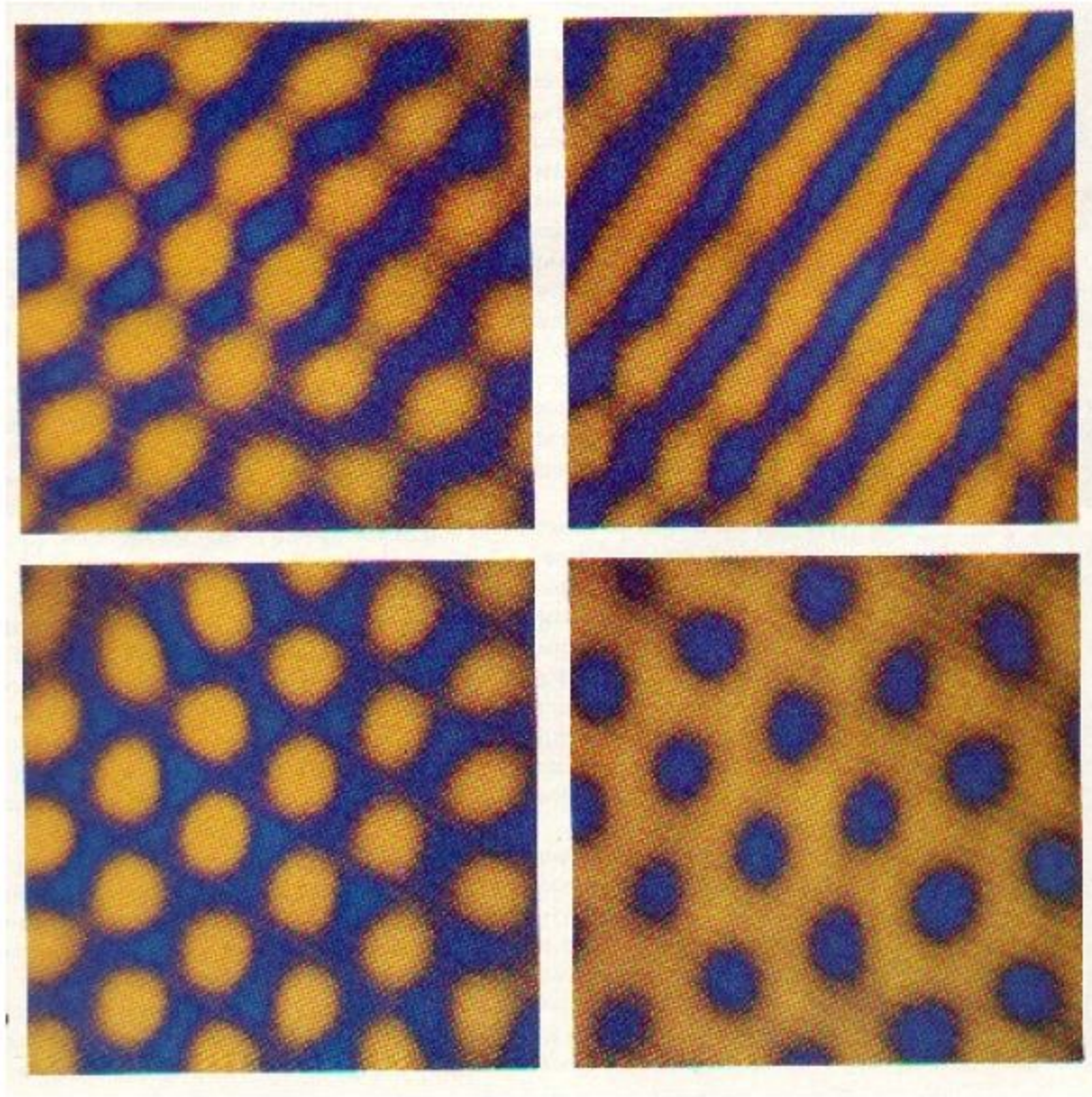
- detJ jest dane przez $a_{11}a_{22} – a_{12}a_{21}$; już wiemy ze $a_{11}a_{22} < 0$
- Żeby detJ było dodatnie musi $a_{12}a_{21} < 0$
- Tylko dwa macierze Jacobiego się kwalifikują jako wzorce Turinga:

$$\begin{pmatrix} + & + \\ - & - \end{pmatrix} \qquad \begin{pmatrix} + & - \\ + & - \end{pmatrix}$$

Typy wzorców Turinga



J. D. Murray, Mathematical Biology II: Spatial Models and Biomedical Applications (Springer, 2003)



Activator-Inhibitor model

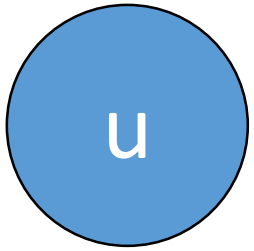
- $\frac{\partial u}{\partial t} = \frac{u^2}{v} - bu + D_u \frac{\partial^2 u}{\partial x^2}$

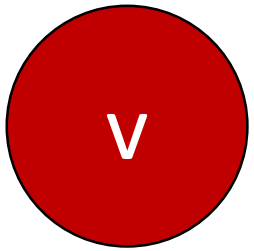
- $\frac{\partial v}{\partial t} = u^2 - v + D_v \frac{\partial^2 v}{\partial x^2}$

Activator-Inhibitor model

- $\frac{\partial u}{\partial t} = \frac{u^2}{v} - bu + D_u \frac{\partial^2 u}{\partial x^2}$

- $\frac{\partial v}{\partial t} = u^2 - v + D_v \frac{\partial^2 v}{\partial x^2}$

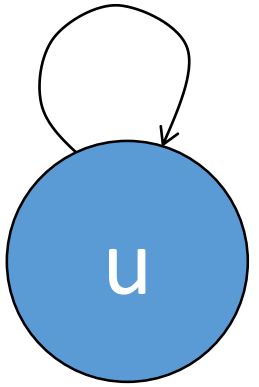


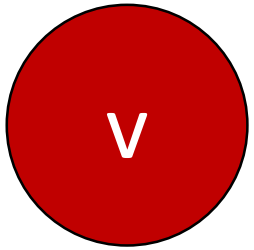


Activator-Inhibitor model

- $\frac{\partial u}{\partial t} = \frac{u^2}{v} - bu + D_u \frac{\partial^2 u}{\partial x^2}$

- $\frac{\partial v}{\partial t} = u^2 - v + D_v \frac{\partial^2 v}{\partial x^2}$

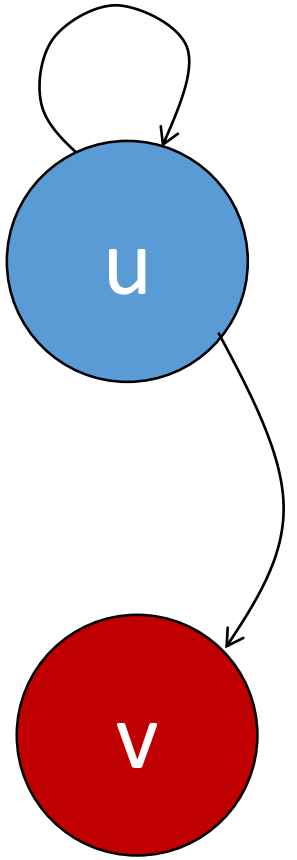




Activator-Inhibitor model

- $\frac{\partial u}{\partial t} = \frac{u^2}{v} - bu + D_u \frac{\partial^2 u}{\partial x^2}$

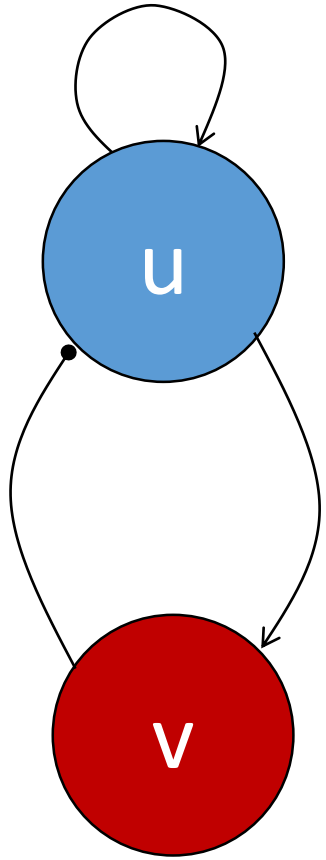
- $\frac{\partial v}{\partial t} = u^2 - v + D_v \frac{\partial^2 v}{\partial x^2}$



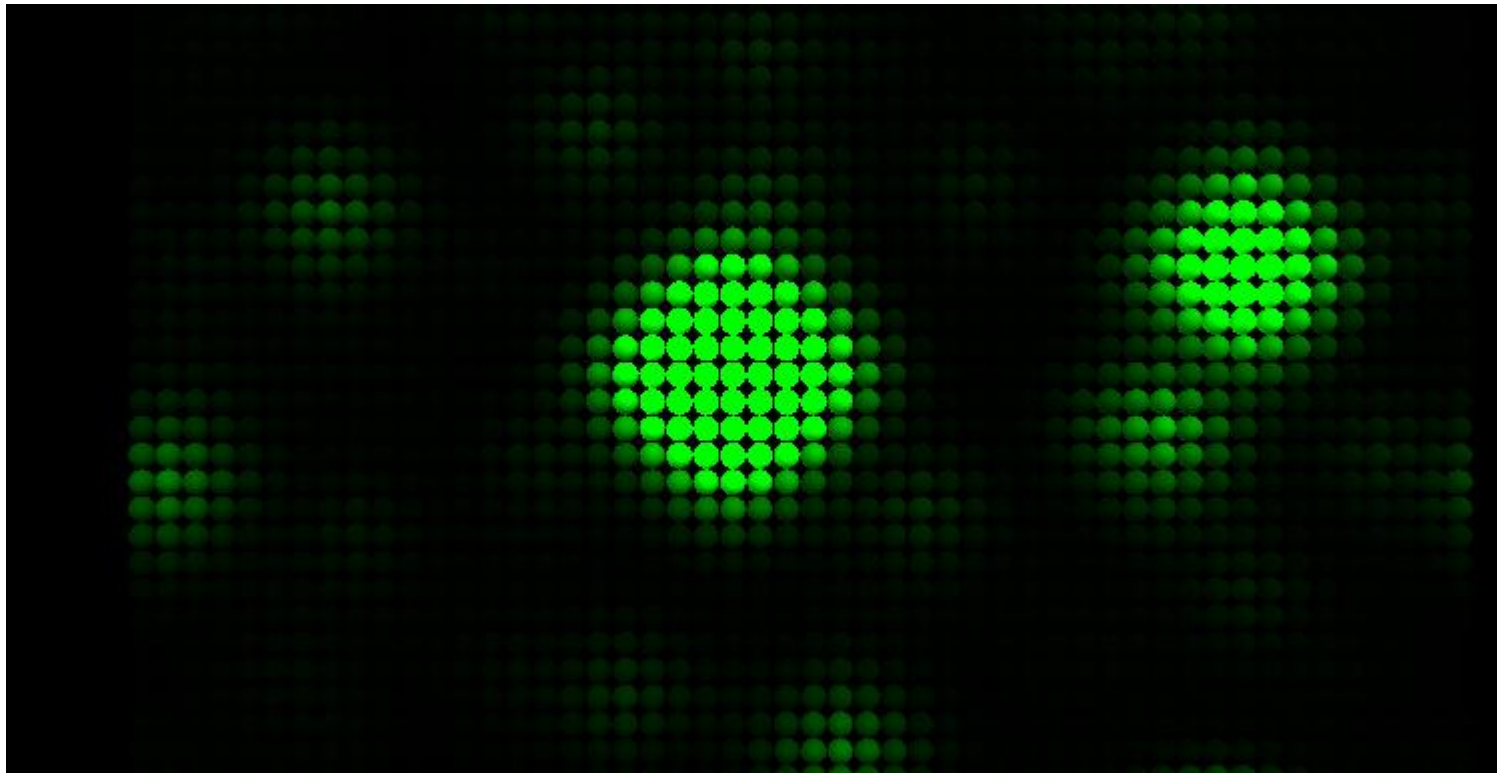
Activator-Inhibitor model

- $\frac{\partial u}{\partial t} = \frac{u^2}{v} - bu + D_u \frac{\partial^2 u}{\partial x^2}$

- $\frac{\partial v}{\partial t} = u^2 - v + D_v \frac{\partial^2 v}{\partial x^2}$

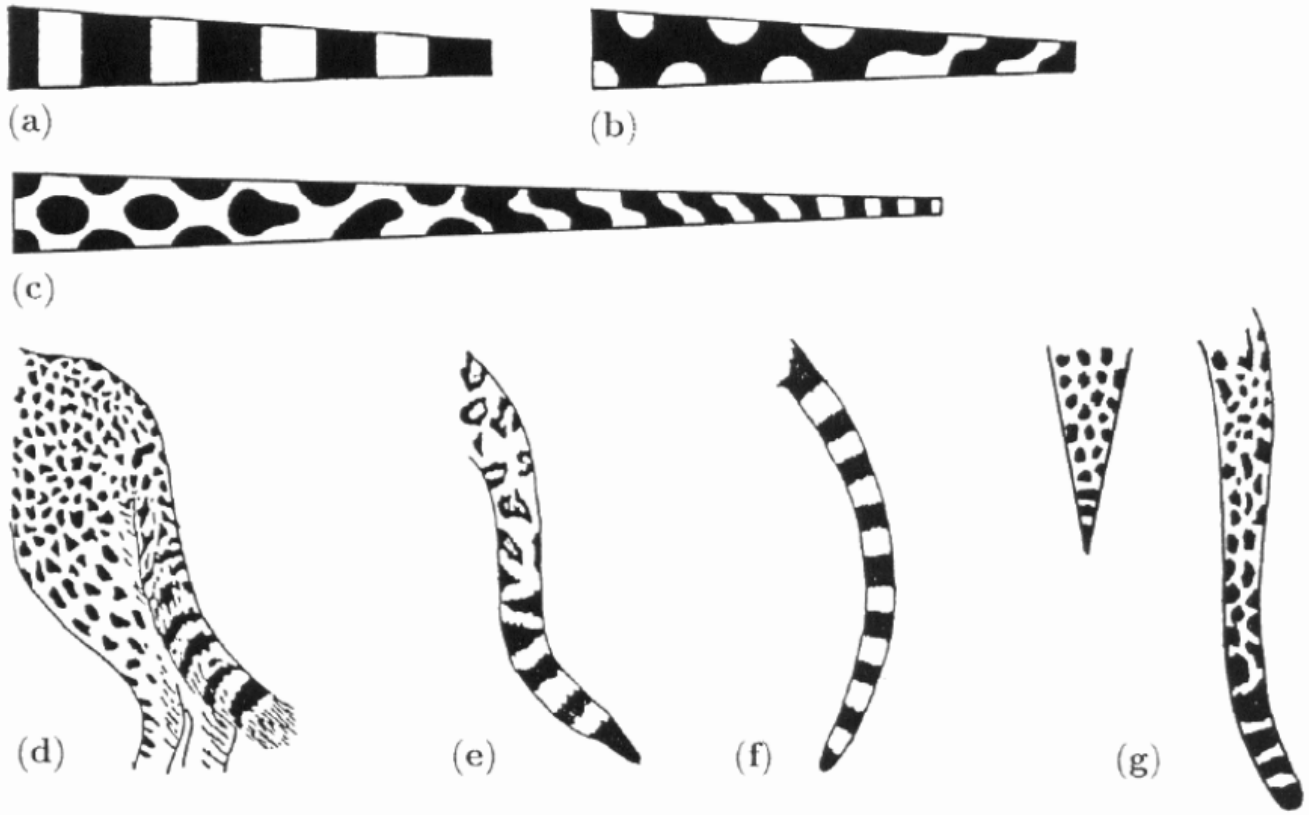


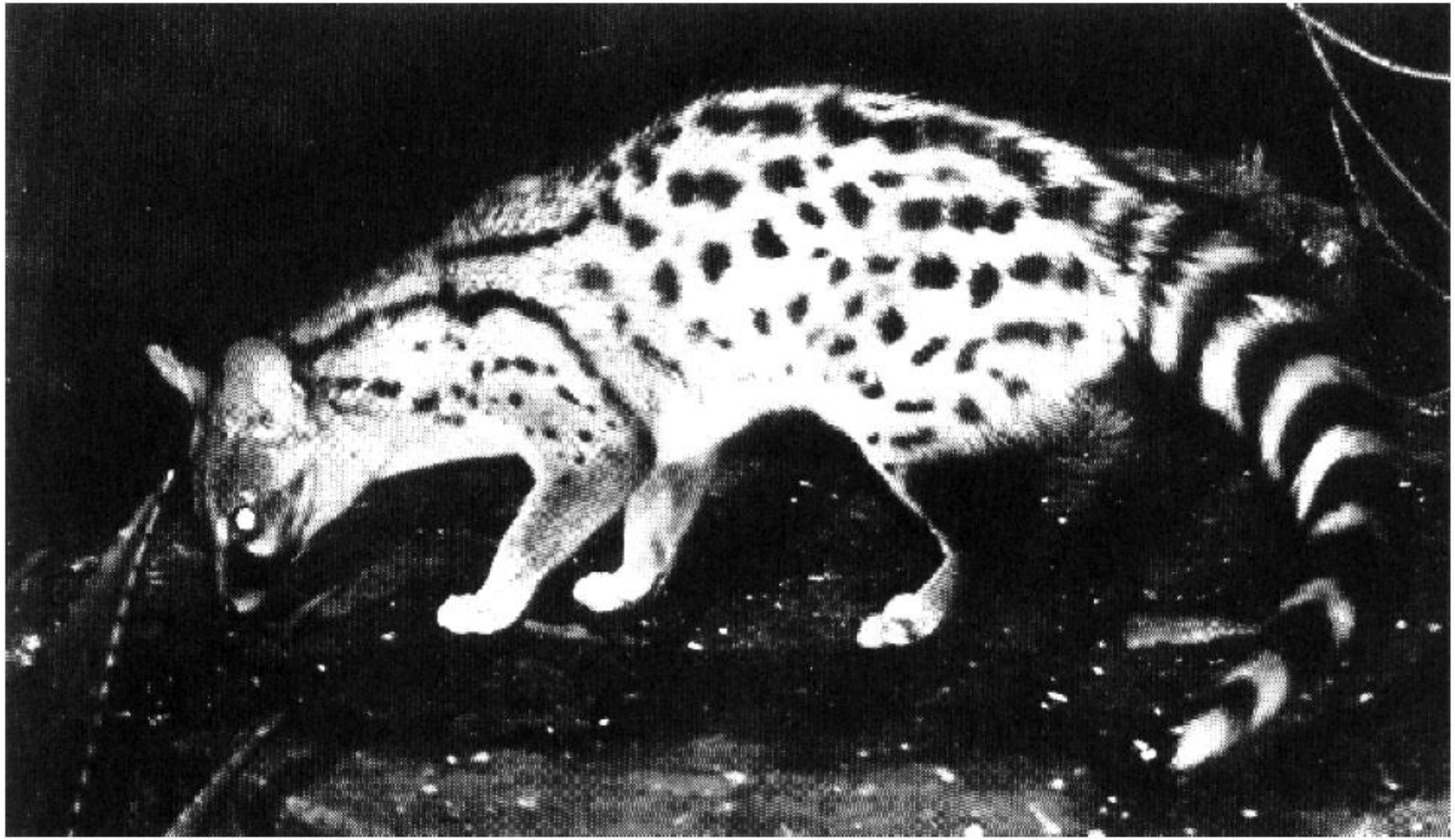
Activator-Inhibitor 2D



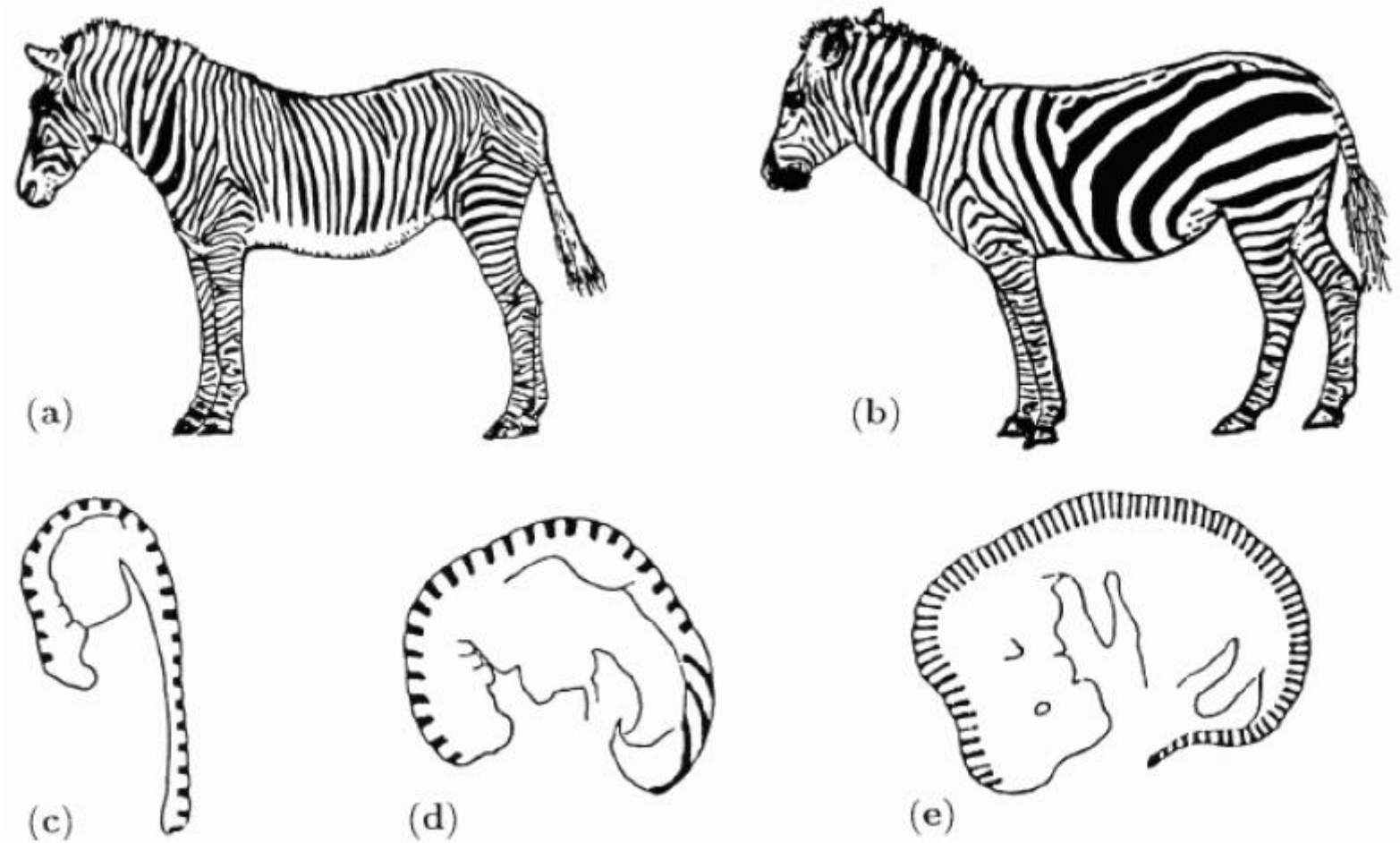
Activator-Inhibitor 2D

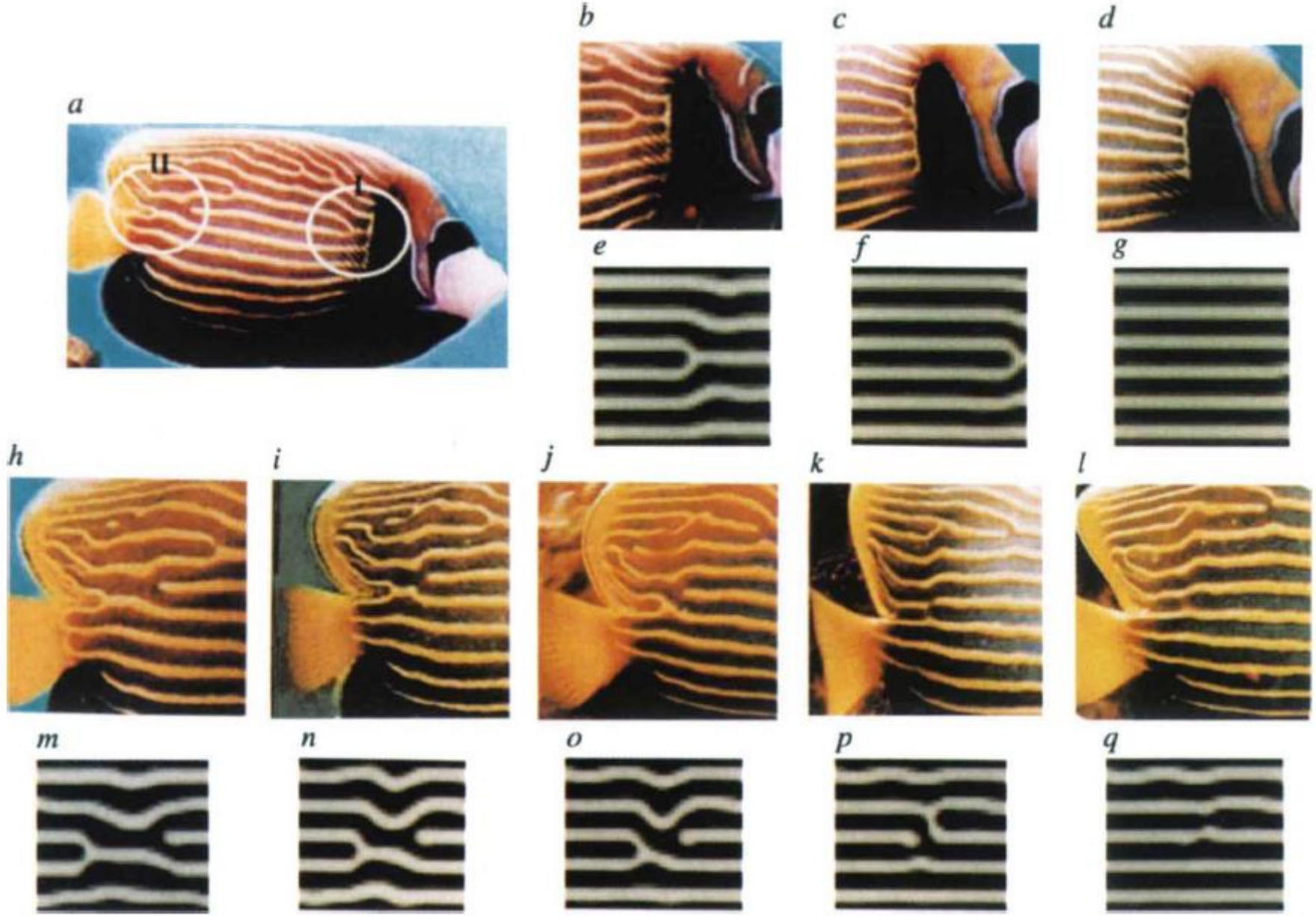
Ogony kotów






Scapular
stripes
(a)
(b)


(a)
(b)
(c)
(d)
(e)



Kondo S. et al, Nature 1995

Zadanie

- Zaimplementuj model aktywatora-inhibitora.